Motion of vortex lines in quantum mechanics

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\textbf{Abstract} In quantum theory, vortex lines arise in the hydrodynamic interpretation of the wave equation. In this interpretation, which is originally due to Madelung, the flow of the probability density for a single particle is described in terms of the hydrodynamic variables. For the sake of simplicity, the standard time-dependent Schrödinger equation, and the related vortex lines embedded in the probability fluid of the quantum particle, are considered here. A vortex line in this case is simply the curve defined by equating the wave function to zero. The linearity of the Schrödinger equation enables us to obtain a large family of exact time-dependent analytic solutions for the wave functions with vortex lines. Moreover, the method is general enough to allow for various initial configurations of the vortex lines. Although the equation of motion of the quantum mechanical probability fluid is different in its literal form from the equations describing the real physical fluid, we believe that the evolution of the vorticity in the quantum and in the real fluid share the same qualitative features that can be described in terms of the topology of the vortex lines configurations. The general phenomena such as the switchover, creation and annihilation of vortices can be observed in the quantum mechanical fluid.
1. Introduction

The principle of probability conservation in non-relativistic quantum mechanics gives rise to a hydrodynamic interpretation of the quantum wave equations. If we consider the Schrödinger equation for a quantum particle as an example, the probability density is $|\psi(r, t)|^2$, and there is a simple formula for the probability flow current. Therefore, it is valid to interpret the probability density as the density of some “quantum fluid”, and the corresponding conserved current as the flow of this “quantum fluid”. In Bialynicki-Birula et. al. (2000) it was argued that the zeros of a generic solution of the linear Schrödinger equation correspond to vortex line singularities in the quantum flow.

In section 2 the objects of the study are defined. In section 3 it is shown that the property of vorticity quantisation follows directly from the definition of the hydrodynamic variables. A very simple example of a straight vortex is given in section 4. A general method of generating more complex vortex solutions of linear wave equations is presented in section 5.

2. Basic definitions and current conservation in quantum mechanics

The probability density $\rho(r, t)$, the conserved current $j(r, t)$ and the velocity field $v(r, t)$ are connected with the wave function $\psi(r, t)$ through the formulas

$$ j(r, t) = \rho(r, t) v(r, t) $$
$$ \psi(r, t) = R(r, t) \exp[iS(r, t)/\hbar], $$
$$ \rho(r, t) = |\psi(r, t)|^2 = R^2(r, t), $$
$$ v(r, t) = \frac{1}{m} \frac{\text{Re}[\psi^*(r, t)(-i\hbar\nabla - eA)\psi(r, t)]}{|\psi(r, t)|^2} $$
$$ = \frac{1}{m}(\nabla S(r, t) - eA(r, t)), $$
where \( m \) is the mass of the particle and \( \mathbf{A}(\mathbf{r}, t) \) is the electromagnetic vector potential. The continuity equation reads

\[
\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r}, t) = 0. \tag{5}
\]

Since the denominator \( |\psi(\mathbf{r}, t)|^2 \) in equation (4) may have zeros, the velocity field \( \mathbf{v}(\mathbf{r}, t) \) may have singularities like in case of a vortex line in a real fluid. In general, the zeros of a complex function of three real variables are one-dimensional varieties, i.e. lines. In the present context they are vortex lines embedded in the solution \( \psi(\mathbf{r}, t) \) of the Schrödinger equation

\[
\frac{i\hbar}{\hbar} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}) \right] \psi(\mathbf{r}, t). \tag{6}
\]

### 3. Quantisation of vortex strength

Since in the standard quantum mechanics the wave function is always a single valued function of the coordinates, the strength of a vortex line as measured by the circulation \( \Gamma \) along any closed contour \( C \) encircling the vortex line and not intersecting other vortex lines,

\[
\Gamma = \oint_C d\mathbf{l} \cdot \mathbf{v}(\mathbf{r}, t), \tag{7}
\]

must be quantised:

\[
\Gamma = \frac{2\pi \hbar}{m} n, \quad n = 0, \pm 1, \pm 2, \ldots \tag{8}
\]

Indeed, the phase \( S(\mathbf{r}, t)/\hbar \) is well defined (up to a constant and modulo \( 2\pi \), though), and the circulation \( \Gamma \) is the total change of the phase along the contour \( C \).

If \( n \neq 0 \), the velocity \( \mathbf{v}(\mathbf{r}, t) \) must tend to infinity as one approaches the vortex line, so as to satisfy the quantisation condition (8). Hence, nonzero vortex strength implies a singularity of the velocity field: quantum vortices are line singularities.

### 4. An example: a straight vortex line

To give a simple example of a vortex line let us consider the wave function

\[
\psi(x, y, z) = x + iy. \tag{9}
\]

The vortex line equation \( \psi(x, y, z) = 0 \) defines a straight vortex line along the \( z \)-axis (intersection of the planes \( x = 0 \) and \( y = 0 \)). In this case the vortex strength \( n = 1 \).
5. A general method of generating vortex solutions

Let us choose as the initial condition for the Schrödinger equation an element of the continuous family of wave functions

\[ \phi_k(r) = \exp(i k \cdot r) \phi_0(r), \] (10)

parameterised be the components of the wave vector \( k \). It is assumed that a solution \( \psi_k(r, t) \) of the time-dependent Schrödinger equation is known that corresponds to the initial condition \( \psi_k(r, t = 0) = \phi_k(r) \). By differentiating \( \phi_k \) with respect to the components of \( k \) (any number of times), we obtain new wave functions. Each differentiation brings down a component of the position vector and in this manner we may generate an arbitrary complex polynomial that multiplies the initial wave function. Carrying out the differentiations, adding the results with appropriate complex coefficients, and setting \( k = 0 \) at the end, we arrive at the expression for the initial wave function of the form

\[ (W_R(r) + iW_I(r))\phi_0(r), \] (11)

where \( W_R \) and \( W_I \) are real polynomials in the three variables \( x, y, \) and \( z \).

Conversely, given a function \( \phi_0(r) \) and two real polynomials \( W_R(r) \) and \( W_I(r) \), we define two differential operators \( W_R(-i \partial_k) \) and \( W_I(-i \partial_k) \) by substituting \((-i \frac{\partial}{\partial x}, -i \frac{\partial}{\partial y}, -i \frac{\partial}{\partial z})\) in place of the variables \( x, y, z \) in the polynomials \( W_R(r) \) and \( W_I(r) \). The two operators, acting on the family \( \psi_k(r, t) \), at \( t = 0 \) yield the polynomial factors \( W_R(r) \) and \( W_I(r) \) again, providing a solution of the Schrödinger equation for the given initial wave function \((W_R(r) + iW_I(r))\phi_0(r)\).

It is an obvious observation that \((W_R(r) + iW_I(r))\) is the polynomial part of the initial wave function, while the remaining part, denoted \( \phi_0(r) \), enters the solution via the initial condition \( \phi_k(r) \).

6. Examples and conclusions

In section 5, a general method was presented to generate vortex solutions of the wave equations. Using this general method, example solutions were obtained for the free particle case, i.e. \( A(r, t) = 0, V(r) = 0 \). The pictures show the particularly peculiar phases of motion of two (fig. 1) and three (fig. 2–5) vortex lines, see the references for details.

Our conclusion is that interesting phenomena occurring in the quantum mechanical probability fluid — in contrast with the “physical” fluids — can be easily described and visualised.
Figure 1. A switchover of two vortex lines embedded in a solution of the free Schrödinger equation. At $t = 0$ there are two non-intersecting straight vortex lines.

Figure 2. Time evolution of vortex lines embedded in a solution of the free Schrödinger equation. At $t = 0$ there are three non-intersecting straight vortex lines in mutually orthogonal directions.
Figure 3. The same solution in a shorter time-scale.

Figure 4. The same solution, $t$ goes from 0.30 to 0.34.

Figure 5. A chosen frame from the previous picture.
References