The Wigner functional of the electromagnetic field

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Abstract

Wigner’s idea to describe a quantum-mechanical system in terms of an almost classical distribution function in the phase space is extended to the case of the full electromagnetic field. The role of positions and momenta is played by the magnetic and electric induction vectors and the analog of the Wigner function is a functional of $B$ and $D$. Properties of the Wigner functionals for thermal states, coherent states, many-photon states, and squeezed states are discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Wigner function introduced almost 70 years ago [1,2] to calculate quantum corrections to a classical distribution function of a quantum-mechanical system has recently become a very popular tool to study the dynamics of photons (cf., for example, [3,4]). There will even be the whole monograph devoted to the phase-space methods in quantum optics [5]. In these studies the electromagnetic field is described in the one-mode approximation and the photons are treated as excitations of a one-dimensional, quantum-mechanical harmonic oscillator. In this paper I proceed differently following our earlier studies of squeezing phenomena in space-time [8–11]. The whole electromagnetic field is treated as one huge, infinitely-dimensional harmonic oscillator. The wave function and the corresponding Wigner function become then functionals of the field variables. Still, the connection with the one-dimensional oscillator is so close that everything that I shall need can be done by analogy with this very simple case. The global approach presented here enables one to study the dynamics of the quantized electromagnetic field in space and time and not just the quantum properties of a single mode with a fixed mode function. The Wigner functional in field theory has been discussed before by Mrowczynski and Muller [6] but they have only considered a scalar field which has much less structure as compared to the electromagnetic field.

1 This work is dedicated to Marlan Scully on the occasion of his 60th birthday.
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2. Wigner functional

It has been shown a long time ago by Wheeler [7] that the vacuum state of the quantized electromagnetic field can be described by the wave functional in SI units

\[ \Psi_0[A] = C \exp \left[ -\frac{1}{4\pi^2\hbar} \sqrt{\frac{e}{\mu}} \int \int d^3r d^3r' B(r) \frac{1}{|r - r'|^2} \cdot B(r') \right], \]

which is the lowest energy eigenstate of the Hamiltonian of the electromagnetic field

\[ H = \int d^3r \left[ -\frac{\hbar^2}{2\varepsilon} \frac{\delta^2}{\delta A(r)^2} + \frac{1}{2\mu} (\nabla \times A(r))^2 \right]. \]

The determination of the normalization constant \( C \) is a complex mathematical issue (see the discussion at the end of this section) but even with an unknown \( C \) one can calculate all relative probabilities. In this representation the canonically conjugate variables are the vector potential \( A \) and the (minus) electric displacement vector \(-D\). The former is represented by the multiplication by \( A \) and the latter by the functional derivative

\[ \hat{D}(r) = i\hbar \frac{\delta}{\delta A(r)}. \]

The minus sign is chosen to have the same sign in the commutation relations between the position and momentum as in ordinary quantum mechanics. The functional (1) is the counterpart of the ground-state wave function

\[ \psi(x) = (m\omega/\pi\hbar)^{1/4} \exp(-x^2m\omega/2\hbar) \]

of a harmonic oscillator in the Schrödinger (position) representation. Owing to the full symmetry of the free Maxwell theory under the interchange of electricity and magnetism, there exist also the representation of the ground state in the form [11]

\[ \tilde{\Psi}_0[\tilde{A}] = C \exp \left[ -\frac{1}{4\pi^2\hbar} \sqrt{\frac{e}{\mu}} \int \int d^3r d^3r' D(r) \frac{1}{|r - r'|^2} \cdot D(r') \right], \]

where \( \tilde{A} \) is the vector potential for the \( D \) vector, \( D = \nabla \times \tilde{A} \). This wave functional is the counterpart of the ground-state wave function in momentum representation

\[ \tilde{\psi}(p) = (\hbar m\omega\pi)^{-1/4} \exp(-p^2/2m\omega\hbar). \]

Note that the canonically conjugate variables in quantum electrodynamics are different in those two representations; in one case it is \((A, -D)\) and in the other it is \((\tilde{A}, \tilde{B})\).

The Wigner function for the ground state of the one-dimensional harmonic oscillator is

\[ W(x, p) = \frac{1}{\pi\hbar} \exp \left( -\frac{p^2/m + m\omega^2x^2}{\hbar\omega} \right) \].
This expression is obtained from the definition of the Wigner function in the $N$-dimensional configuration space \cite{1,2}

\[
W(r,p) = (2\pi\hbar)^{-N} \int d^N\eta \exp(i\eta \cdot p/\hbar)\psi(r - \eta/2)\psi^*(r + \eta/2),
\]

(8)

upon the substitution of the ground state wave function (4) into (8) and after the evaluation of the integral. In order to find the Wigner functional of the ground state of the electromagnetic field let us note that the exponent in the formula (7) is equal to twice the classical energy of the oscillator divided by the energy per one quantum $\hbar\omega$, i.e. to twice the average number of quanta. The average number of quanta (photons) in electrodynamics can be obtained by the same prescription, namely by dividing the total field energy by the energy of the quantum. Since the energy of the photon depends on its wave vector, the division must be performed after the Fourier transformation. This operation is most easily done with the use of the Riemann-Silberstein complex vector $\mathbf{F}$ that features so prominently in the study of the photon wave function \cite{14–16}

\[
\mathbf{F}(r,t) = \frac{\mathbf{D}(r,t)}{\sqrt{2}\epsilon} + \frac{\mathbf{B}(r,t)}{\sqrt{2}\mu}.
\]

(9)

In the present case, the convenience of using this vector comes from the fact that one does not have to impose cumbersome reality constraints on the Fourier transforms. The field energy $\mathcal{E}[\mathbf{A},\mathbf{D}]$ expressed in terms of $\mathbf{F}$ and its Fourier transform $\hat{\mathbf{F}}$ is

\[
\mathcal{E}[\mathbf{A},\mathbf{D}] = \int d^3r \mathbf{F}^*(r,t) \cdot \mathbf{F}(r,t) = \int d^3k \hat{\mathbf{F}}^*(k,t) \cdot \hat{\mathbf{F}}(k,t).
\]

(10)

The average number of quanta $\mathcal{N}[\mathbf{A},\mathbf{D}]$ is

\[
\mathcal{N}[\mathbf{A},\mathbf{D}] = \int d^3k \frac{\hat{\mathbf{F}}^*(k,t) \cdot \hat{\mathbf{F}}(k,t)}{\hbar\omega_k} = \frac{1}{2\pi^2\hbar c} \int d^3r \int d^3r' \mathbf{F}^*(r,t) \frac{1}{|r-r'|^2} \cdot \mathbf{F}(r',t),
\]

(11)

where I have used the relation

\[
\int \frac{d^3k}{\hbar\omega_k} e^{ik \cdot (r-r')} = \frac{4\pi}{\hbar c |r-r'|^2}.
\]

(12)

Returning to the standard notation, employing the field vectors $\mathbf{B}$ and $\mathbf{D}$, one obtains

\[
\mathcal{N}[\mathbf{A},\mathbf{D}] = \frac{1}{4\pi^2\hbar} \int d^3r \int d^3r \left( \sqrt{\frac{\epsilon}{\mu}} B(r) \frac{1}{|r-r'|^2} \cdot B(r') + \sqrt{\frac{\mu}{\epsilon}} D(r) \frac{1}{|r-r'|^2} \cdot D(r') \right).
\]

(13)

This expression for the average number of photons in the electromagnetic field has been derived long time ago by Zeldovich \cite{12}. It also plays the role of a norm for the photon wave function \cite{13–16}. This functional of the electromagnetic field vectors has some remarkable properties. Despite its 'nonrelativistic' appearance it is
invariant not only under the Poincaré group but also under the full conformal group \cite{1963}. It is also manifestly gauge invariant. Using the formula \cite{1963}, one arrives at the following expression for the Wigner functional of the electromagnetic field in the ground state

$$W_0[A, D] = \exp(-2 N[A, D])$$

$$= \exp\left[-\frac{1}{2\pi^2\hbar} \int \! d^3r \int \! d^3r' \left( \sqrt{\frac{e}{\mu}} B(r) \frac{1}{|r - r'|} \cdot B(r') + \sqrt{\frac{\mu}{\epsilon}} D(r) \frac{1}{|r - r'|} \cdot D(r') \right) \right].$$

(14)

My normalization of the Wigner functional is different from the standard normalization of the Wigner function: I have assumed a pure exponential form without a factor $1/\pi\hbar$ per each degree of freedom that was present in the formula \cite{1962}. In the infinitely dimensional case such a change of normalization is forced upon us by the divergence of the infinite product of such factors. The problem of normalization would be automatically solved by giving a rigorous definition of the integration measure in the functional phase space of $B$ and $D$ but I shall not address here this subtle mathematical issue. Without the resolution of this problem one may still calculate all relative probabilities.

The Wigner functional \cite{1964} is positive definite. Therefore it can be interpreted (apart from the unknown overall normalization) as a genuine probability distribution of the field vectors $B$ and $D$. We learn from the formula \cite{1964} a few facts about the ground state of the electromagnetic field in free space or in a homogeneous medium.

- Field correlations have a long range; they die out only as the inverse square of the distance. Thus, quantum electrodynamics is intrinsically nonlocal.
- The direction of the field vectors oscillates violently; it is much more likely to find the directions of the two field vectors to be opposite than to find them pointing in the same direction and this anticorrelation increases at small separations with the inverse distance squared.
- Increasing the dielectric/magnetic properties of the medium leads to the flattening of the distribution of the values of $D/B$ and to the peaking of the distribution of the values of $B/D$.
- The Wigner functional gives a description of the electromagnetic field that is fully symmetric under the exchange of electricity and magnetism.

3. Thermal state of the electromagnetic field

The Wigner function for the thermal state $W_\gamma(x, p)$ of a one-dimensional harmonic oscillator is well known (see, for example, [2,17])

$$W_\gamma(x, p) = \frac{\tanh(\hbar\omega/2kT)}{\pi\hbar} \exp\left(-\tanh(\hbar\omega/2kT) \frac{p^2}{m + \hbar^2\omega^2 x^2} \right).$$

(15)

The Wigner functional for the thermal state of the electromagnetic field can be found with the use of the same analogy that lead us from the formula \cite{1962} to \cite{1964}. The coordinate representation of the factor $\tanh(\hbar\omega/2kT)/\hbar\omega$ in momentum space has the following representation in coordinate space

$$\int \frac{d^3k}{\hbar\omega_k} e^{ik \cdot (r-r')} \tanh(\hbar\omega_k/2kT) = \frac{8\pi^2kT}{\hbar^2c^2 |r - r'| \sinh(2\pi kT |r - r'|/\hbar c)}. \quad (16)$$
The Wigner functional for the thermal state is obtained from (14) by substituting for \(4\pi/\hbar c|\mathbf{r} - \mathbf{r}'|\) the new expression (16). In the limit, when \(T \to 0\), both expressions coincide. Since the thermal state is described by a positive functional, one may again treat it as a probability distribution and use it to study statistical properties of the field. Correlations of the electromagnetic field vectors in the thermal state, in contradistinction to the vacuum state, involve the characteristic length parameter \(l_T = \hbar c/\kappa T\) that determines the range of these correlations. They do not have a long range anymore but they die out exponentially with the decay length \(l_T\). For example, at the room temperature 300 K the correlation length is equal to \(7.6 \times 10^{-6}\) m. Thus, at this temperature, thermal fluctuations wipe out all correlations and at macroscopic distances electromagnetic field values become statistically independent.

4. Coherent states and their superposition

In quantum mechanics the wave function of the coherent state characterized by the mean position \(\mathbf{X}\) and the mean momentum \(\mathbf{P}\) is

\[
\psi(x) = (m\omega/\pi\hbar)^{1/4}\exp\left[-m\omega(x - \mathbf{X})^2/2\hbar\right]\exp[i\mathbf{P}(x - \mathbf{X}/2)/\hbar]. \tag{17}
\]

The corresponding Wigner function \(W_{\text{coh}}(x,p)\) has the form of the vacuum Wigner function displaced in phase space by \(\mathbf{X}\) and \(\mathbf{P}\)

\[
W_{\text{coh}}(x,p) = \frac{1}{\pi\hbar}\exp\left(-\frac{(p - \mathbf{P})^2/m + m\omega^2(x - \mathbf{X})^2}{\hbar\omega}\right). \tag{18}
\]

The state functional for the electromagnetic field in the coherent state and the corresponding Wigner functional can be written down by analogy with the formulas (17) and (18). I shall denote by \(\mathbf{A}\) and \(\mathbf{D}\) the vector potential and the electric displacement vectors – the counterparts of \(\mathbf{X}\) and \(\mathbf{P}\) – that characterize the coherent state of the electromagnetic field. The vector potential, as in the study of the ground state, will enter through the magnetic induction vector \(\mathbf{B}\). In order to obtain the coherent state of the field one has to displace the vacuum functionals (1) and (14) by these values of the field vectors. In addition, the state vector of the coherent state must be multiplied by the appropriate phase factor as in the formula (17). This phase factor can again be obtained by the replacement

\[
\mathbf{P}\mathbf{X} \to -\int d^3r \mathbf{D}(r) \cdot \mathbf{A}(r). \tag{19}
\]

Since the divergence of \(\mathbf{D}\) vanishes in the free case, the longitudinal (gauge) part of the vector potential does not contribute and the transverse part can be expressed by \(\mathbf{B}\), resulting in an explicitly gauge invariant expression

\[
\int d^3r \mathbf{D}(r) \cdot \mathbf{A}(r) = \frac{1}{4\pi} \int d^3r \int d^3r' \left(\mathbf{D}(r) \cdot \frac{1}{|r - r'|^2} \nabla \times \mathbf{D}(r')\right). \tag{20}
\]
In this way one arrives at the following formulas for the state functional of the coherent state

\[
\Psi_{\text{coh}}[A] = C \exp \left[ - \frac{1}{4\pi^2 \hbar} \sqrt{\frac{\epsilon}{\mu}} \int d^3r \int d^3r' \frac{(B - B') \cdot (B' - B')}{|r - r'|^2} \right] \exp \left[ - \frac{i}{4\pi \hbar} \int d^3r \cdot (A - \mathcal{A}/2) \right]
\]  

(21)

and the Wigner functional of the coherent state

\[
W_{\text{coh}}[A, D] = \exp \left[ - \frac{1}{2\pi^2 \hbar} \int d^3r \int d^3r' \left( \sqrt{\frac{\epsilon}{\mu}} \frac{(B - B') \cdot (B' - B')}{|r - r'|^2} + \sqrt{\frac{\mu}{\epsilon}} \frac{(D - D') \cdot (D' - D')}{|r - r'|^2} \right) \right]
\]  

(22)

where the non-primed field vectors depend on \( r \) and the primed ones depend on \( r' \).

Now we have all the ingredients needed for the discussion of the superposition of coherent state of the electromagnetic field. The Wigner function \( W_{\text{sup}}(x,p) \) of the superposition \( \psi_{\text{sup}} = \alpha \psi_1 + \beta \psi_2 \) of two states is

\[
W_{\text{sup}} = |\alpha|^2 W_1 + |\beta|^2 W_2 + 2 \text{Re}(\alpha \beta^* W_{12}),
\]

(23)

where \( W_1 \) and \( W_2 \) are the Wigner functions of the two states and \( W_{12} \) is the off-diagonal Wigner function (a.k.a. the Moyal function [18]) describing the interference effects,

\[
W_{12}(r,p) = (2\pi \hbar)^{-N} \int d^N \eta \exp(i \eta \cdot p / \hbar) \psi_1(r - \eta/2) \psi_2^*(r + \eta/2).
\]

(24)

For the two coherent states of the form (17) the off-diagonal Wigner function \( W_{12} \) is

\[
W_{12}(x,p) = \frac{1}{\pi \hbar} \exp \left( - (p - P_+)^2 / m \hbar \omega + m \omega (x - \mathcal{X}_+)^2 / \hbar \right) \exp \left( i (x - \mathcal{X}_+ / 2) P_- - (p - P_+ / 2) \mathcal{X}_- / \hbar \right),
\]

(25)

where I have introduced the ‘center of mass’ variables (labeled with +) and the relative variables (labeled with -),

\[
\mathcal{X}_+ = (\mathcal{X}_1 + \mathcal{X}_2) / 2, \quad P_+ = (P_1 + P_2) / 2,
\]

\[
\mathcal{X}_- = (\mathcal{X}_1 - \mathcal{X}_2), \quad P_- = (P_1 - P_2).
\]

(26)

The off-diagonal Wigner functional \( W_{12}[A, D] \) for two coherent states obtained by analogy from (25) is

\[
W_{12}[A, D] = \exp \left[ - \frac{1}{2\pi^2 \hbar} \int d^3r \int d^3r' \left( \sqrt{\frac{\epsilon}{\mu}} \frac{(B - B_+ \cdot (B' - B'_+)}{|r - r'|^2} + \sqrt{\frac{\mu}{\epsilon}} \frac{(D - D_+ \cdot (D' - D'_+)}{|r - r'|^2} \right) \right]
\]

\[
\times \exp \left[ - \frac{i}{\hbar} \int d^3r (A - \mathcal{A}_+/2) \cdot \mathcal{D}_- - (D - \mathcal{D}_+/2) \cdot \mathcal{A}_- \right].
\]

(27)

The plus and minus combinations of the field vectors are defined as in (26). Thus, the quantum superposition of the two coherent states characterized by the classical field values \((B_1, D_1)\) and \((B_2, D_2)\) gives the quasi-distribution functional (Wigner functional) in phase space that has three peaks: two smooth peaks at the original classical values and one oscillatory peak at half the sum of the two classical fields. On the other hand, the
coherent state representing the classical superposition of the two classical fields gives the Wigner functional that has just one peak at the point in phase space defined by the sums \((\mathcal{R}_1 + \mathcal{R}_2)\) and \((\mathcal{R}_1 + \mathcal{R}_2)\). Therefore, as has been emphasized often for one-mode states (see, for example, \[19,20\]), the quantum and the classical superpositions of coherent fields have a totally different physical meaning.

5. Photon states

Wigner functionals can be also written down for the states of the electromagnetic field that describe a definite number of photons. The simplest such state is the one-photon state. The wave functional describing this state is obtained, as in the case of a harmonic oscillator, by multiplying the vacuum functional by a linear combination of position variables. In quantum mechanics this linear combination would be \(f_{x'} \cdot A\) and in presently studied case of quantum electrodynamics it is \(f_{x'} \cdot A\), where \(f(r)\) is the photon mode function,

\[
\Psi_{1ph}[A] = \int \! d^3r f(r) \cdot A(r) \exp \left[ -\frac{1}{4\pi^2\hbar} \sqrt{\frac{\epsilon}{\mu}} \int \! d^3r \int \! d^3r' B(r) \frac{1}{|r-r'|^2} \cdot B(r') \right]. \tag{28}
\]

Of course, the photon wave function must be transverse, \(\nabla \cdot f = 0\), to make \(28\) gauge invariant. The one-photon wave functional can be obtained from the functional of the coherent state \(17\) by differentiating it with respect to \(\mathcal{R}_1\), multiplying by \(i\), setting \(\mathcal{R}_1\) and \(\mathcal{R}_2\) equal to zero and integrating with \(f_{x'}\) over all space. Therefore, a coherent state functional may serve as a generator of \(N\)-photon states. Differentiating this functional \(N\) times with respect to \(\mathcal{R}_1\) produces the product of \(N\) vectors \(A\). This means that the off-diagonal Wigner function may serve as a generating functional for all \(N\)-photon Wigner functionals: the Wigner functional for many photon states is obtained from the functional \(27\) by simple differentiation without having to do any functional integrations. In particular, the Wigner functional for the one-photon state is

\[
W_{1ph}[B,D] = \hbar^2 \int \! d^3r f(r) \cdot \frac{\delta}{\delta \mathcal{R}_1(r)} \int \! d^3r f(r) \cdot \frac{\delta}{\delta \mathcal{R}_2(r)} W_{1ph}[B,D] \bigg|_{\mathcal{R}_1=0,\mathcal{R}_2=0}
\]

\[
= \left( -\frac{1}{2\pi^2} \sqrt{\frac{\mu}{\epsilon}} \int \! d^3r \int \! d^3r' \frac{f \cdot D'}{|r-r'|^2} \right)^2 + \int \! d^3r f \cdot A \right)^2 - \frac{\hbar}{\pi^2} \sqrt{\frac{\mu}{\epsilon}} \int \! d^3r \int \! d^3r' \frac{f \cdot f'}{|r-r'|^2}
\]

\[
\times \exp \left[ -\frac{1}{2\pi^2\hbar} \int \! d^3r \int \! d^3r' \left( \sqrt{\frac{\epsilon}{\mu}} B(r) \frac{1}{|r-r'|^2} \cdot B(r') + \sqrt{\frac{\mu}{\epsilon}} D(r) \frac{1}{|r-r'|^2} \cdot D(r') \right) \right]. \tag{29}
\]

One may recognize in this expression the same general structure as in the quantum-mechanical Wigner function of the lowest excited state of the harmonic oscillator

\[
W_{lex}(x,p) = \frac{2}{\pi \hbar^2} \left( \frac{p^2}{m \omega} + m \omega x^2 - \frac{\hbar}{2} \right) \exp \left( -\frac{p^2/m + m \omega^2 x^2}{\hbar \omega} \right). \tag{30}
\]

In both cases the Wigner function becomes negative when the values of the canonical variables become too small. In the quantum-mechanical case the region of negative values is enclosed by the ellipse

\[
\frac{p^2}{m \omega} + m \omega x^2 = \frac{\hbar}{2}. \tag{31}
\]

In quantum electrodynamics this region is enclosed by an ellipsoid with an infinite number of dimensions.
6. Squeezed states

Squeezed states also fit very well into this framework. Since these states have a Gaussian form of the wave functional [8–11], they can be described by two real kernels $K_r$ and $K_I$,

$$
\Psi_{\text{sq}}[A] = C \exp \left[ -\frac{\hbar}{\epsilon} \int \frac{d^3 r}{\mu} \left( K_r(r) \cdot \left( B(r) \cdot K_I(r') + i K_I(r) \cdot B(r') \right) \right) \right].
$$

that replace the inverse of the squared distance found in the formula (14). The Wigner functional will also be of a Gaussian form. The most general such functional can be parametrized by three real matrix kernels $K_{BB}, K_{DD},$ and $K_{BD},$

$$
W_{\text{sq}}[B,D] = \exp \left( -\frac{1}{\hbar} \int d^3 r_1 \int d^3 r_2 \left[ \sqrt{\frac{\epsilon}{\mu}} B \cdot K_{BB} \cdot B' + \sqrt{\frac{\mu}{\epsilon}} D \cdot K_{DD} \cdot D' + B \cdot K_{BD} \cdot D' \right] \right).
$$

Only the purely transverse part of each kernel $K$ contributes in this formula because the field vectors are transverse. Not all functionals of this general form represent allowed physical states because there are only two (not three!) independent kernels that characterize a Gaussian squeezed state. These kernels must obey the condition related to the Schrödinger-Robertson form of the uncertainty relation [11].

Under the time evolution the Wigner functionals of squeezed states retain their Gaussian form but the kernels $K$ change, in general, with time according to the ‘Maxwell type’ set of coupled linear equations that can be easily derived from Eq. (36). Time dependence of these kernels leads to the propagation of squeezing [10,11].

7. Time evolution of the Wigner functional

Owing to the quadratic form of the potential, all quantum corrections drop out and the Wigner function for the harmonic oscillator obeys the same equations of motion as the classical distribution function,

$$
\partial_t W(r,p,t) = -\left( \frac{p}{m} \cdot \frac{\partial}{\partial r} - m \omega^2 r \cdot \frac{\partial}{\partial p} \right) W(r,p,t).
$$

The solution of the initial value problem for this equation is obtained (see, for example, [21]) by simply inserting the solution of the initial value problem for the classical equations of motion $r(r_0,p_0,t), p(r_0,p_0,t)$ evaluated at $-t$, into the initial Wigner function,

$$
W(r_0,p_0,-t) = W(r(r_0,p_0,0),p(r_0,p_0,0),t = 0).
$$

Since the Maxwell equations are also linear, the same holds true for the Wigner functional. The equation of motion and its solution can be written by analogy with (34) and (35). To this end, one has to observe that the expressions multiplying the position and momentum derivatives in (34) are just the time derivatives of the position and momentum and this leads to the following equation of motion for the Wigner functional (there is a wrong sign on the right hand side of this equation in my previous paper [11])

$$
\partial_t W[B,D,t] = \int d^3 r \left( \frac{\nabla \times D(r)}{\epsilon} \cdot \frac{\delta}{\delta B(r)} - \frac{\nabla \times B(r)}{\mu} \cdot \frac{\delta}{\delta D(r)} \right) W[B,D,t].
$$
The solution of this equation can also be written in terms of the solution $B[B_0, D_0|t], D[B_0, D_0|t]$ of the initial value problem for the Maxwell equations

$$W[B_0, D_0|t] = W[B[B_0, D_0|t], D[B_0, D_0|t], t = 0].$$

(37)

In the special case of the coherent state (22) one may use the invariance of the vacuum state under the time evolution to transfer the time evolution from the arguments of the Wigner functional $B$ and $D$ to the labels $BB$ and $DD$ of the coherent state and obtain the time-dependent Wigner functional in the form

$$W_{coh}(B, D|t) = \exp \left[ -\frac{1}{2\pi \hbar} \int d^3r \int d^3r' \left( \sqrt{\frac{e}{\mu}} \frac{(B - BB(t)) \cdot (B' - BB'(t))}{|r - r'|^2} \right. \right.$$

$$\left. + \sqrt{\frac{\mu}{\epsilon}} \frac{(D - DD(t)) \cdot (D' - DD'(t))}{|r - r'|^2} \right) \right].$$

(38)

where $BB(t)$ and $DD(t)$ are solutions of Maxwell equations. Thus, as one might have expected, the center of the Wigner functional evolves in phase space according to the classical Maxwell equations.

8. Conclusions

I have shown in this paper how to extend the formalism of the Wigner function to quantum electrodynamics of the free electromagnetic field. We have seen that this extension can be easily done just by following the analogy between the electromagnetic field and the quantum-mechanical harmonic oscillator. As is always the case with Wigner function formalism, it is not a very powerful tool for making detailed calculations but it often enables us to make qualitative estimates and it offers us additional insights into the workings of quantum theories.

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References