

Rényi Entropy and the Uncertainty Relations

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Abstract. Quantum mechanical uncertainty relations for the position and the momentum and for the angle and the angular momentum are expressed in the form of inequalities involving the Rényi entropies. These uncertainty relations hold not only for pure but also for mixed states. Analogous uncertainty relations are valid also for a pair of complementary observables (the analogs of x and p) in N -level systems. All these uncertainty relations become more attractive when expressed in terms of the symmetrized Rényi entropies. The mathematical proofs of all the inequalities discussed in this paper can be found in Phys. Rev. A **74**, No. 5 (2006); arXiv:quant-ph/0608116.

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1. CRITICAL ANALYSIS OF THE STANDARD UNCERTAINTY RELATIONS

The celebrated uncertainty relation $\Delta x \Delta p \geq \hbar/2$ envisioned by Heisenberg [1] and proved by Kennard [2] is a trademark of quantum mechanics. However, the statement made by Heisenberg “The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa” can be given many different mathematical forms. The formulation of the uncertainty principle with the help of standard deviations is only one possibility.

The standard deviation is used a lot in the statistical analysis of experiments. It is a reasonable measure of the spread or localization when the probability distribution is of a simple "single hump" type. For example, it is a very good characteristic for a Gaussian distribution since it measures the half-width of this distribution. However, when the probability distribution has more than one hump, the standard deviation loses some of its usefulness, especially in connection with the notion of uncertainty. In some cases it gives results that contradict common sense. Sometimes for reasonable physical states the standard deviation is infinite.

In order to explain, why the standard deviation is not the best measure of uncertainty, I shall repeat my arguments presented some time ago [3]. Let us consider two very simple examples taken from quantum mechanics of a particle moving in one dimension. In the first example let us compare two states of a particle. One state describes the particle localized with a uniformly distributed probability in a box of length L and the other describes the particle localized with equal probabilities in two smaller boxes each of length $L/4$.



FIGURE 1. The particle is localized with a uniformly distributed probability in a box of length L .

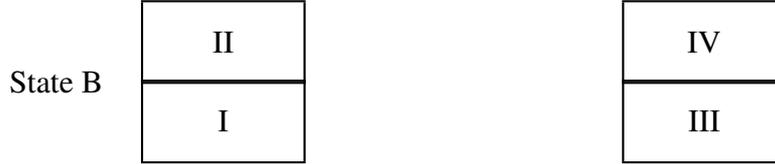


FIGURE 2. The particle is localized with equal probabilities in two smaller boxes each of length $L/4$.

The probability distributions corresponding to these two states are graphically represented as shown in Fig. 1 and Fig. 2.

Before continuing let us ponder in which case, A or B, the uncertainty concerning the position of the particle is greater. According to common sense, the uncertainty is greater in the case A. In the case B we know more about the position; we know that the particle is not in the regions II and III. However, when we calculate the standard deviation Δx we obtain the opposite result:

$$\text{Case A} \quad \Delta x_A = \frac{L}{\sqrt{12}}, \quad (1)$$

$$\text{Case B} \quad \Delta x_B = \sqrt{\frac{7}{4}} \frac{L}{\sqrt{12}}. \quad (2)$$

The second, somewhat more dramatic example of the situation where the standard deviation does not give a sensible measure of uncertainty is provided by the following distribution of probability. The wave function does not vanish and is constant in two regions I and II separated by a large distance NL (N is a large number). The region I is of the size $L(1 - 1/N)$ and the region II is of the size L/N (Fig. 3).

$$\text{State C} \quad \psi = \begin{cases} \frac{1}{\sqrt{L}} & \text{in region I,} \\ \frac{1}{\sqrt{L}} & \text{in region II,} \\ 0 & \text{elsewhere.} \end{cases} \quad (3)$$

For large N the standard deviation Δx is approximately equal to:

$$\text{Case C} \quad \Delta x_C \sim \sqrt{NL}, \quad (4)$$

so that it tends to infinity with N in spite of the fact that the probability of finding the particle in the region I tends to 1. This example shows most vividly what is wrong with the standard deviation. It gets very high contributions from distant regions, because these

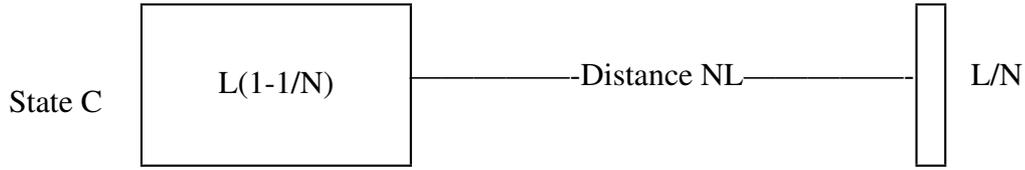


FIGURE 3. The particle is to be found mostly in a central region. In addition, there is a small probability that the particle can be found far away.

contribution enter with a large weight: the distance from the mean value squared. I have discussed the position of the particle but the same criticism applies to the standard deviation used as a measure of the uncertainty for momentum or other physical quantities. In order to illustrate once again the fact that the weighing of the probability density with the distance squared overemphasizes the contributions from distant regions, let me to consider the standard deviation for the momentum variable corresponding to the wave function describing the state A in my first example. The momentum representation of this wave function is

$$\tilde{\psi}(p) = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \exp(ipx/\hbar) = \frac{2\hbar \sin(pL/2\hbar)}{\sqrt{L} p}. \quad (5)$$

The probability density for the momentum in the state described by this wave function when p tends to infinity falls off only as $1/p^2$. Therefore, the second moment of this distribution is infinite and the Heisenberg uncertainty relation expressed in terms of the standard deviations becomes meaningless.

Can we overcome the deficiencies of the traditional approach to uncertainty relations by changing the measure of uncertainty while still keeping the spirit of the Heisenberg ideas intact? Not only is the answer to this question in the affirmative but we can do it using a very profound definition of uncertainty, much more fundamental than the one based on the second moment. Such definition comes from information theory. Let us take a look at uncertainty from the point of view of information. What is uncertainty? It is just the lack of information; a hole in our knowledge or missing information. Therefore, the uncertainty can be measured in exactly the same manner as the information is measured. The widely accepted measure of information is the one introduced by Shannon. There are also generalizations due to Rényi and Tsallis. In this paper, I shall deal with the measure of uncertainty introduced by Rényi.

2. THE RÉNYI ENTROPY AS A MEASURE OF UNCERTAINTY

The Rényi entropy is a one-parameter generalization of the Shannon entropy. There is extensive literature on the applications of the Rényi entropy in many fields from biology, medicine, genetics, linguistics, and economics to electrical engineering, computer science, geophysics, chemistry and physics. My aim is to describe the limitations on

the information characterizing quantum systems, in terms of the Rényi entropies. These limitations have the form of inequalities that have the physical interpretation of the uncertainty relations.

The Rényi entropy has been widely used in the study of quantum systems. In particular, it was used in the analysis of quantum entanglement [4, 5, 6, 7], quantum communication protocols [8, 9], quantum correlations [10], quantum measurement [11] and decoherence [12], multi-particle production in high-energy collisions [13, 14, 15], quantum statistical mechanics [16], localization properties of Rydberg states [17] and spin systems [18, 19], in the study of the quantum-classical correspondence [20], and the localization in phase space [21]. In view of these numerous and successful applications, it seems worthwhile to formulate the quantum-mechanical uncertainty relations for canonically conjugate variables in terms of the Rényi entropies. I do not want to enter here into the discussion (cf. [22, 23]) of a fundamental problem: which (if any) entropic measure of uncertainty is adequate in quantum mechanical measurements. The uncertainty relations discussed in this paper are valid as mathematical inequalities, regardless of their physical interpretation.

The Rényi entropy is defined [24] as

$$H_\alpha = \frac{1}{1-\alpha} \ln \left(\sum p_k^\alpha \right). \quad (6)$$

Rényi called this quantity “the measure of information of order α associated with the probability distribution $\mathcal{P} = (p_1, \dots, p_n)$ ”. The Rényi measure of information H_α may also be viewed as a measure of uncertainty since after all the uncertainty is the missing information. In the formulation of the uncertainty relations given below, the Rényi entropies will be used as measures of uncertainties. In order to simplify the derivations, I use the natural logarithm in the definition (6) of the Rényi entropy. However, all uncertainty relations discussed in this paper (Eqs. (13), (16), (19), (23), (24), (26), (27), and (28)) have the same form for all choices of the base of the logarithm because they are homogeneous in $\ln(\dots)$. Note that the definition of the Rényi entropy is also applicable when the sum has infinitely many terms, provided this infinite sum converges. The Rényi entropy (6) is a decreasing function of α . In the limit, when $\alpha \rightarrow 1$ it is equal (apart from a different base of the logarithm) to the Shannon entropy

$$\lim_{\alpha \rightarrow 1} H_\alpha = - \sum p_k \ln p_k. \quad (7)$$

In the case C of the previous section that led to an infinite value of the dispersion, the Rényi entropy for the position is

$$H_\alpha = \frac{1}{1-\alpha} \ln \left((1-1/N)^\alpha + (1/N)^\alpha \right) \sim \frac{1}{1-\alpha} \left((1/N)^\alpha - \alpha/N \right). \quad (8)$$

This quantity, when $N \rightarrow \infty$, tends to 0 as it should because the uncertainty tends to 0.

3. THE FORMULATION OF THE UNCERTAINTY RELATIONS IN TERMS OF THE RÉNYI ENTROPIES

According to the probabilistic interpretation of quantum theory, the probability distribution associated with the measurement of a physical variable represented by the operator A is defined as

$$p_k = \text{Tr}\{\rho P_k\}, \quad (9)$$

where ρ is the density operator describing the state of the quantum system, and P_k is the projection operator corresponding to the k -th segment of the spectrum of A (the k -th bin). The uncertainty is the lowest when only one p_k is different from zero — the Rényi entropy reaches then its lowest value: zero. The probability distributions p_k^A and p_k^B that correspond to different physical variables but to the same state of the system are, in general, correlated. These correlations lead to restrictions on the values of the Rényi entropies H_α^A and H_β^B . When these restrictions have the form of an inequality $H_\alpha^A + H_\beta^B \geq C > 0$, they deserve the name of the uncertainty relations because not only they prohibit the vanishing of *both* uncertainties for the same state but they also require that one uncertainty must increase when the other decreases.

In the present paper, I describe the inequalities for three pairs of observables: position and momentum (or time and frequency), angle and angular momentum, and the complementary observables — the analogs of x and p — in finite dimensional spaces. These inequalities are generalizations of the entropic uncertainty relations established before for the Shannon entropies [25, 26, 27].

4. RÉNYI ENTROPIES FOR POSITION AND MOMENTUM

The probability distributions associated with the measurements of momentum and position for a pure state (generalization to mixed states is discussed in Sec. 8) are

$$p_k = \int_{k\delta_p}^{(k+1)\delta_p} dp |\tilde{\psi}(p)|^2, \quad q_l = \int_{l\delta_x}^{(l+1)\delta_x} dx |\psi(x)|^2, \quad (10)$$

where I have assumed that the sizes of all bins are the same. The indices k and l run from $-\infty$ to ∞ and the Fourier transform is defined with the physical normalization, i.e.

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx e^{-ipx/\hbar} \psi(x). \quad (11)$$

From the two probability distributions (10) we may construct the Rényi entropies $H_\alpha^{(p)}$ and $H_\beta^{(x)}$ that measure the uncertainty in momentum and position

$$H_\alpha^{(p)} = \frac{1}{1-\alpha} \ln \left(\sum p_k^\alpha \right), \quad H_\beta^{(x)} = \frac{1}{1-\beta} \ln \left(\sum q_l^\beta \right). \quad (12)$$

5. UNCERTAINTY RELATIONS FOR x AND p

The uncertainty relation restricting the values of $H_\alpha^{(p)}$ and $H_\beta^{(x)}$ has the following form

$$H_\alpha^{(p)} + H_\beta^{(x)} \geq -\frac{1}{2} \left(\frac{\ln \alpha}{1-\alpha} + \frac{\ln \beta}{1-\beta} \right) - \ln \left(\frac{\delta x \delta p}{\pi \hbar} \right), \quad (13)$$

where the parameters α and β are assumed to be positive and they are constrained by the relation

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2. \quad (14)$$

In the limit, when $\alpha \rightarrow 1$ and $\beta \rightarrow 1$, this uncertainty relation reduces to the uncertainty relation for the Shannon entropies

$$H^{(p)} + H^{(x)} \geq -\ln \left(\frac{\delta x \delta p}{e\pi\hbar} \right) \quad (15)$$

that has been already derived some time ago [27]. The proof of the inequalities (13) and (15) requires some, fairly sophisticated properties of the Fourier transforms and is given in [29].

Note that the relations (13) and (15) are quite different from the standard uncertainty relations. As has been aptly stressed by Peres [30], “The uncertainty relation such as $\Delta x \Delta p \geq \hbar/2$ is not a statement about the accuracy of our measuring instruments”. In contrast, both entropic uncertainty relations (13) and (15) *do depend* on the accuracy of the measurement — they explicitly contain the area of the phase-space $\delta x \delta p$ determined by the resolution of the measuring instruments. These relations can be summarized as follows: the more precisely one wants to localize the particle in the phase space, the larger the sum of uncertainties in x and p . The uncertainty relation (13) is not sharp — its improvement is a challenging open problem. However, it becomes sharper and sharper when the relative size of the phase space area $\delta x \delta p / \pi \hbar$ defined by the experimental resolutions decreases, it is when we enter deeper and deeper into the quantum regime.

6. UNCERTAINTY RELATIONS FOR φ AND M_z

The uncertainty relations in terms of the Rényi entropies can also be formulated for the angle φ and the angular momentum M_z and they have the form

$$H_\alpha^{(M_z)} + H_\beta^{(\varphi)} \geq -\ln \frac{\delta \varphi}{2\pi}. \quad (16)$$

The probability distributions $p_m^{(M_z)}$ and $p_l^{(\varphi)}$ that are used to calculate these Rényi entropies are defined as follows

$$p_m^{(M_z)} = |c_m|^2, \quad p_l^{(\varphi)} = \int_{l\delta\varphi}^{(l+1)\delta\varphi} d\varphi |\psi(\varphi)|^2, \quad (17)$$

where the amplitudes c_m are the coefficients in the expansion of $\psi(\varphi)$ into the eigenstates of M_z ,

$$\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} c_m e^{im\varphi}, \quad (18)$$

and $\delta\varphi$ is the experimental resolution in the measurement of the angular distribution. In contrast to the uncertainty relation for position and momentum, the inequality (16) is sharp (it is saturated by any eigenstate of M_z) and the bound does not depend on α and β . The absence of the Planck in this uncertainty relation is due to a cancellation — the volume of the phase space defined by the experimental resolution is $\delta\varphi \delta M_z = \delta\varphi \hbar$ and the standard reference volume in quantum theory is $2\pi\hbar$.

The uncertainty relations (16) and (27) also hold for the phase and the occupation number of a mode of radiation. In this case, the Fourier expansion (18) contains only the terms with $m \geq 0$.

7. UNCERTAINTY RELATIONS FOR N -LEVEL SYSTEMS

For quantum systems described by vectors in the N -dimensional Hilbert space the analog of the uncertainty relation for the Rényi entropies is

$$\frac{1}{1-\alpha} \ln \left(\sum_{k=1}^N \tilde{\rho}_k^\alpha \right) + \frac{1}{1-\beta} \ln \left(\sum_{l=1}^N \rho_l^\beta \right) \geq \ln N, \quad (19)$$

where $\tilde{\rho}_k = |\tilde{a}_k|^2$, $\rho_l = |a_l|^2$ and the amplitudes \tilde{a}_k and a_l are connected by the discrete Fourier transformation

$$\tilde{a}_k = \frac{1}{\sqrt{N}} \sum_{l=1}^N \exp(2\pi i k l / N) a_l. \quad (20)$$

The complex numbers \tilde{a}_k and a_l can be interpreted as the probability amplitudes to find a particle in the discretized momentum space and position space [31], but they can also be viewed as amplitudes in a general abstract N -dimensional Hilbert space. The uncertainty relation (19) is saturated for the states that are localized either in “position space” (only one of the amplitudes a_l is different from zero) or in “momentum space” (only one of the amplitudes \tilde{a}_k is different from zero). Like in the case of the uncertainty relations for the angle and the angular momentum, the bound does not depend on α and β . The absence of the Planck constant is again due to a cancellation — it would reappear if l and k in (20) are given the physical dimension of length and momentum.

8. UNCERTAINTY RELATIONS FOR MIXED STATES

The uncertainty relations for the Rényi entropies hold also for all mixed states. This result is not obvious because the Rényi entropy is not a convex function [28] of the

probability distributions for all values of α . Hence, the terms on the left hand side of the uncertainty relation (13) may decrease as a result of mixing. Therefore, as explained in detail in [29], one has to go back to the original inequalities that were used to prove (13) and show that they still hold for mixed states.

9. THE UNCERTAINTY RELATIONS FOR CONTINUOUS DISTRIBUTIONS

There also exist purely mathematical versions of the uncertainty relations that *do not involve* the experimental resolutions δx and δp of the measuring devices. They have the form

$$\begin{aligned} & \frac{1}{1-\alpha} \ln \left(\int_{-\infty}^{\infty} dp (\tilde{\rho}(p))^\alpha \right) + \frac{1}{1-\beta} \ln \left(\int_{-\infty}^{\infty} dx (\rho(x))^\beta \right) \\ & \geq -\frac{1}{2(1-\alpha)} \ln \frac{\alpha}{\pi} - \frac{1}{2(1-\beta)} \ln \frac{\beta}{\pi}. \end{aligned} \quad (21)$$

On the left hand side of this inequality we have, what might be called, the continuous or integral versions of the Rényi entropies. I have dropped \hbar in the definition (11) of the Fourier transform in order to derive this purely mathematical inequality that includes no reference to the finite resolution of the physical measurements. This inequality has been also recently independently proven by Zozor and Vignat [33]. Analogous relations for the continuous Tsallis entropies for x and p were obtained by Rajagopal [34]. In the limit, when $\alpha \rightarrow 1$, $\beta \rightarrow 1$, we obtain from the formula (21) the entropic uncertainty relation in the form

$$-\int_{-\infty}^{\infty} dp \tilde{\rho}(p) \ln \tilde{\rho}(p) - \int_{-\infty}^{\infty} dx \rho(x) \ln \rho(x) \geq \ln(e\pi) \quad (22)$$

that had been conjectured by Hirschman [35] and later proved by Bialynicki-Birula and Mycielski [36] and by Beckner [37].

In a similar fashion we can introduce the uncertainty relation for φ and M_z that does not involve the resolution $\delta\varphi$.

$$\frac{1}{1-\alpha} \ln \left(\sum_{-\infty}^{\infty} \rho_m^\alpha \right) + \frac{1}{1-\beta} \ln \left(\int_0^{2\pi} d\varphi (\rho(\varphi))^\beta \right) \geq \ln(2\pi), \quad (23)$$

where $\rho_m = |c_m|^2$ and $\rho(\varphi) = |\psi(\varphi)|^2$. In the limit, when $\alpha \rightarrow 1$ and $\beta \rightarrow 1$, we obtain the mathematical entropic uncertainty relation for the angle and the angular momentum derived before [36]

$$-\sum_{-\infty}^{\infty} \rho_m \ln \rho_m - \int_0^{2\pi} d\varphi \rho(\varphi) \ln \rho(\varphi) \geq \ln(2\pi). \quad (24)$$

The inequalities (23) and (24), like their discrete counterpart (16), are saturated when the Fourier series (18) has only one term.

10. SYMMETRIZED RÉNYI ENTROPIES

In the uncertainty relations for the Rényi entropies the parameters α and β appear always in conjugate pairs. This observation suggests the introduction of the *symmetrized Rényi entropy* \mathcal{H}_s defined as follows

$$\mathcal{H}_s = \frac{1}{2} (H_\alpha + H_\beta), \text{ where } \alpha = \frac{1}{1-s}, \beta = \frac{1}{1+s}, \quad -1 \leq s \leq 1. \quad (25)$$

The symmetrized Rényi entropy \mathcal{H}_s is a symmetric function of s and for $s = 0$ it becomes the Shannon entropy. The uncertainty relations expressed in terms of the symmetrized Rényi entropies have the form

$$\mathcal{H}_s^{(p)} + \mathcal{H}_s^{(x)} \geq \frac{1}{2} \left(\ln(1-s^2) + \frac{1}{s} \ln \frac{1+s}{1-s} \right) - \ln \left(\frac{\delta x \delta p}{\pi \hbar} \right). \quad (26)$$

They are obtained by taking half of the sum of the inequality (13) and the inequality obtained from (13) by interchanging α and β . The same symmetrization procedure can be applied to all other uncertainty relations discussed in this paper. In particular, we can obtain

$$\mathcal{H}_s^{(M_z)} + \mathcal{H}_s^{(\varphi)} \geq -\ln \frac{\delta \varphi}{2\pi}. \quad (27)$$

Analogous symmetrized versions of the uncertainty relations for the Tsallis entropies were introduced also by Rajagopal [34].

In contrast to the inequalities that contain the Rényi entropies H_α and H_β , in the uncertainty relations that contain the symmetrized entropy the *same measure of uncertainty* is used for both physical variables. This is clearly a desirable feature but it remains to be seen whether the symmetrized Rényi entropy (25) is a useful concept outside the realm of the uncertainty relations.

Different uncertainty relations in which the same measure of uncertainty is used for both variables follow from the Rényi entropy being a nonincreasing function of α . For example, for the position and momentum we obtain

$$H_\beta^{(p)} + H_\beta^{(x)} \geq -\ln \beta - \frac{\beta - 1/2}{1 - \beta} \ln(2\beta - 1) - \ln \left(\frac{\delta x \delta p}{\pi \hbar} \right), \quad (28)$$

where $1 \geq \beta \geq 1/2$.

11. CONCLUSIONS

I have shown that quantum mechanical uncertainty relations for canonically conjugate variables can be expressed as inequalities involving the Rényi entropies. The simplicity of these relations indicates, in my opinion, that the Rényi entropy is an apt characteristic of the uncertainties in quantum measurements. A significant feature of these relations is the appearance of the resolving power of the measuring apparatus. Since the Rényi

entropy is an extension of the Shannon entropy, the new uncertainty relations generalize the entropic uncertainty relations derived before. The formulation of the uncertainty relations in terms of the Rényi entropies seems to indicate that a symmetrized version of the Rényi entropy (25) might be a useful concept.

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