

Nonlinear Wave Mechanics*

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Nonlinear wave mechanics is constructed, based on Schrödinger-type equation with nonlinearity $-b\psi \ln |\psi|^2$. This nonlinearity is selected by assuming the factorization of wavefunctions for composed systems. Its most attractive features are: existence of the lower energy bound and validity of Planck's relation $E = \hbar\omega$. In any number of dimensions, soliton-like solutions (gaussons) of our equation exist and move in slowly varying fields like classical particles. The Born interpretation of the wavefunction is consistent with logarithmic nonlinearity and we tentatively estimate the order of magnitude of the universal constant b .

1. INTRODUCTION

All linear equations describing the evolution of physical systems are known to be approximations to some nonlinear theories, with only one notable exception of the Schrödinger equation:

$$i\hbar(\partial/\partial t)\Psi = H\Psi. \quad (1.1)$$

Previous attempts to construct a nonlinear theory which would replace the standard version of quantum mechanics have always run into serious problems of interpretation and correspondence to ordinary linear theory. The most elaborate program to create an intrinsically nonlinear wave mechanics has been developed

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by de Broglie and his collaborators [1]. This was a very ambitious program aimed at creating a causal underlying nonlinear structure basically different from the linear theory. The linear theory was to describe only the statistical behavior of that new structure. No specific nonlinear equation, however, has emerged from those investigations.

The purpose of this paper is to outline and realize a much more modest program. We will show that there exists a nonlinear version of the Schrödinger equation, the one with logarithmic nonlinearity, possessing several attractive features. This equation can be used to describe quantum phenomena without substantial changes in the interpretation of the wavefunction.

We believe that the very fact that there exists nonlinear wave mechanics, capable of describing all known experimental facts, may be of some significance; especially in view of recent growing interest in nonlinear theories. Various nonlinear differential equations have been studied during the last few years in search for solutions describing solitons, i.e., stable, shape-preserving, localized waves [3, 4]. The nonlinear Schrödinger equation with logarithmic nonlinearity possesses solitonlike solutions of the gaussian shape (we call them gaussons) in any number of dimensions. These solutions in our nonlinear wave mechanics describe the propagation of wave packets of freely moving particles. The fundamental difference between linear wave mechanics and the nonlinear theory is that in our theory wave packets of freely moving particles do not spread. Moreover, for macroscopic bodies the gaussons describing the center of mass motion are extremely small and move along classical trajectories.

Our version of nonlinear wave mechanics predicts the existence of an intermediate region (mesoworld), which separates the microworld from the macroworld and locates the position of this region on the scale of dimensions. Also, it determines the limits of validity of linear wave mechanics.

In Section 2 we study the properties of a general class of nonlinear wave equations from which we select, with the help of arguments presented in Section 3, the wave equation with logarithmic nonlinearity. In Sections 4, 5, and 6 we study various aspects of our nonlinear equation: general properties of the energy, stationary states and motion, and internal oscillations of gaussons. Finally, in Section 7 we discuss the physical interpretation of wave mechanics with logarithmic nonlinearity. We show in this Section that the new universal constant b (measuring the strength of the nonlinearity in our theory), which was arbitrary in preceding sections, can not exceed an upper limit of 4×10^{-10} eV set by the existing agreement between the linear theory and experiment. We also give a plausibility argument that b actually has a value not much different from this upper limit.

The theory discussed in this paper is nonrelativistic. However, the wave equation with logarithmic nonlinearity can be replaced by a nonlinear Klein–Gordon equation (see [2]), which also has solutions in the form of (relativistic) gaussons.

2. CLASS OF NONLINEAR SCHRÖDINGER-TYPE EQUATIONS

We will study here Schrödinger-type equations with the simplest form of the nonlinearity—the one not involving derivatives of the wavefunction. More precisely we will consider a class of nonlinear wave equations of the form

$$i\hbar(\partial/\partial t) \psi(\mathbf{r}, t) = [-(\hbar^2/2m)\Delta + U(\mathbf{r}, t) + F(|\psi|^2)] \psi(\mathbf{r}, t). \quad (2.1)$$

We will consider only square-integrable solutions of this equation. The argument \mathbf{r} of the wavefunction stands for the position vector of a single particle or for the whole collection of position vectors (position vector in configuration space) of several particles. If these particles have different masses we must rescale their position vectors in order to obtain the same mass in the kinetic energy term. The function F is assumed to be a real-valued function of the argument $\rho(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$ and may be different for different physical systems.

All equations of type (2.1) have the following properties:

1. The Langrangian density has the form

$$\begin{aligned} \mathcal{L}(\mathbf{r}, t) = & \frac{i\hbar}{2} \left(\psi^* \frac{\partial}{\partial t} \psi - \frac{\partial}{\partial t} \psi^* \cdot \psi \right) \\ & - \frac{\hbar^2}{2m} (\nabla\psi)^* \cdot \nabla\psi - U\psi^*\psi - G(|\psi|^2) \psi^*\psi, \end{aligned} \quad (2.2)$$

where G is defined as

$$G(\rho) = \rho^{-1} \int_0^\rho d\rho' F(\rho'). \quad (2.3)$$

2. Invariance under the multiplication by a constant phase factor holds. Multiplication by a phase harmonically dependent on time and an appropriate shift of the potential U leave Eq. (2.1) invariant.

3. In the absence of the potential, invariance under the full Galileo group holds (including space and time reflections), with the wavefunction transforming in the same manner as in the linear theory.

4. All symmetry properties of the wavefunctions with respect to permutations of the coordinates of identical particles are preserved in time.

5. The sum of two solutions whose overlap is negligible is also a solution (weak superposition principle).

6. The norm $\|\psi\| = (\langle\psi|\psi\rangle)^{1/2}$ (but not the scalar product!) is preserved in time.

7. The total energy functional $E[\psi]$,

$$E[\psi] = (\hbar^2/2m) \langle \nabla\psi | \nabla\psi \rangle + \langle U \rangle + \langle G(|\psi|^2) \rangle + C, \quad (2.4)$$

where C is a constant, for static potentials is preserved in time.

The averaging symbol $\langle \rangle$ denotes here and in what follows the integration with $\psi^*\psi$ and division by the norm squared of ψ . For example,

$$\langle U \rangle = \left[\int d^n r \psi^*(\mathbf{r}) \psi(\mathbf{r}) \right]^{-1} \int d^n r \psi^*(\mathbf{r}) \psi(\mathbf{r}) U(\mathbf{r})$$

and also

$$\langle \nabla\psi | \nabla\psi \rangle = \left[\int d^n r \psi^*(\mathbf{r}) \psi(\mathbf{r}) \right]^{-1} \int d^n r (\nabla\psi(\mathbf{r}))^* \cdot \nabla\psi(\mathbf{r}).$$

8. For stationary states, i.e., for states whose wavefunctions are of the form

$$\psi(\mathbf{r}, t) = \exp(-i\omega t) \psi(\mathbf{r}), \quad (2.5)$$

the frequency ω (which is always real) and the energy $E[\psi]$ are related by the formula

$$E[\psi] = \hbar\omega + C + \langle [G(|\psi|^2) - F(|\psi|^2)] \rangle. \quad (2.6)$$

It is worth noting that wavefunctions of stationary states here, as in the linear theory, are the extremal points of the energy functional (for variations preserving the norm).

9. For isolated systems the total momentum and the total angular momentum defined as in the linear theory, are conserved in time.

10. Expressions for the density and the current and the equations which determine their changes in time have the same form as in the linear theory.

$$\rho(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2, \quad (2.7)$$

$$\mathbf{j}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t) = (\hbar/2mi)(\psi^*\nabla\psi - \nabla\psi^* \cdot \psi), \quad (2.8)$$

$$(\partial/\partial t)\rho = -\nabla \cdot \mathbf{j}, \quad (2.9)$$

$$m(\partial/\partial t)j_k = -\nabla_i T_{ik} - \rho \nabla_k U. \quad (2.10)$$

The stress tensor T_{ij} is given by the formula

$$T_{ij} = -\rho[(\hbar^2/4m) \nabla_i \nabla_j \ln \rho + \delta_{ij}(G - F)] + m\rho v_i v_j. \quad (2.11)$$

The stress tensor here, as in the linear theory, is not defined uniquely by Eq.(2.10). Our form (2.11) is based on the following requirement: We take its part containing

derivatives to be the same as the standard version of the stress tensor in the linear theory and we add only a suitable function of ρ .

11. All solutions satisfy the Ehrenfest theorem:

$$m(d/dt)\langle \mathbf{r} \rangle = \langle \mathbf{p} \rangle, \quad (2.12)$$

$$(d/dt)\langle \mathbf{p} \rangle = -\langle \nabla U \rangle. \quad (2.13)$$

12. The tensor of the second moments Q_{ij} ,

$$Q_{ij} = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle \quad (2.14)$$

changes in time according to the equation

$$\begin{aligned} \frac{m}{2} \frac{d^2}{dt^2} Q_{ij} &= \frac{1}{m} \left\langle \left(\frac{\hbar}{i} \nabla_i - \langle p_i \rangle \right) \psi \left| \left(\frac{\hbar}{i} \nabla_j - \langle p_j \rangle \right) \psi \right\rangle \right. \\ &\quad - \frac{1}{2} \langle (x_i - \langle x_i \rangle) \nabla_j U \rangle - \frac{1}{2} \langle (x_j - \langle x_j \rangle) \nabla_i U \rangle \\ &\quad \left. + \delta_{ij} \langle [F(|\psi|^2) - G(|\psi|^2)] \rangle, \end{aligned} \quad (2.15)$$

which follows from Eqs. (2.7)–(2.13). In particular, for stationary states we obtain

$$\frac{\hbar^2}{m} \langle \nabla_i \psi | \nabla_j \psi \rangle = \frac{1}{2} \langle x_i \nabla_j U \rangle + \frac{1}{2} \langle x_j \nabla_i U \rangle + \delta_{ij} (E[\psi] - \hbar\omega - C). \quad (2.16)$$

After taking the trace over indices i and j we get an expression for twice the kinetic energy of the stationary state (virial theorem)

$$(\hbar^2/m) \langle \nabla \psi | \nabla \psi \rangle = [\langle \mathbf{r} \cdot \nabla U \rangle + n(E[\psi] - \hbar\omega - C)]. \quad (2.17)$$

13. Nonlinear equations of the type (2.1) can be generalized in a natural way to the case of multicomponent, say r -component, wavefunctions (ϕ_1, \dots, ϕ_r) . The most symmetric nonlinear term (the only one invariant under the full unitary group $U(r)$) is of the form

$$i\hbar(\partial/\partial t) \phi_i(\mathbf{r}, t) = \left[-(\hbar^2/2m) \Delta + U(\mathbf{r}, t) + F \left(\sum_{j=1}^r |\phi_j|^2 \right) \right] \phi_i(\mathbf{r}, t). \quad (2.18)$$

Multicomponent wavefunctions always appear in the description of spinning particles. The invariance of Eq. (2.18) in that case leads to the invariance under the rotation of spin. The nonlinear term in Eq. (2.18) couples various components of the wavefunction, but the norm of every component separately is preserved in time.

The energy functional corresponding to Eq. (2.18) has the form

$$E[\phi_1, \dots, \phi_r] = \sum_{k=1}^r \left[(\hbar^2/2m) \langle \nabla \phi_k | \nabla \phi_k \rangle + \langle \phi_k | U | \phi_k \rangle + \langle \phi_k | G \left(\sum_{j=1}^r |\phi_j|^2 \right) | \phi_k \rangle \right]. \quad (2.19)$$

In the simplest case, $r = 2$, Eq. (2.18) reduces for $n = 3$ to a nonlinear generalization of the Pauli equation for the spinning electron.

14. Nonlinear equations of the type (2.1) or (2.18) also can be immediately generalized to include the interaction with an external electromagnetic field. As in the linear theory, this is done through the replacement

$$(\hbar/i)\nabla \rightarrow (\hbar/i)\nabla - (e/c) \mathbf{A}(\mathbf{r}, t) \quad (2.20)$$

and the identification of U with the scalar potential $e\varphi(\mathbf{r}, t)$. In the case of the Pauli equation, one should also add the magnetic moment interaction term $-\mu\boldsymbol{\sigma} \cdot \mathbf{B}$.

In the presence of electromagnetic field the properties 1–11 still hold with only few obvious modifications. Namely, in Eqs. (2.2), (2.4), and (2.8) the replacement (2.20) must be carried out and in Eqs. (2.10) and (2.13) the Lorentz force makes its appearance.

$$m(\partial/\partial t) j_k = -\nabla_i T_{ik} + e\rho(E_k + c^{-1}\epsilon_{iklm}v_l B_m), \quad (2.21)$$

$$(d/dt)\langle \mathbf{r} \rangle = \langle \mathbf{v} \rangle, \quad (2.22)$$

$$m(d/dt)\langle \mathbf{v} \rangle = e(\langle \mathbf{E} \rangle + c^{-1}\langle \mathbf{v} \times \mathbf{B} \rangle), \quad (2.23)$$

where \mathbf{v} is now defined by the modified formula (2.8). The replacement (2.20) also guarantees the gauge invariance with the usual transformation properties of the wavefunctions.

3. THE LOGARITHMIC NONLINEARITY

Let us consider a physical system made up of two noninteracting subsystems, i.e., such that

$$U(x_1, \dots, x_n, t) = U_1(x_1, \dots, x_k, t) + U_2(x_{k+1}, \dots, x_n, t). \quad (3.1)$$

It is well known that in the Schrödinger theory we can write a solution of the wave equation for such a system in the form of a product of two solutions for the

subsystems. This means that if subsystems are initially uncorrelated, they will remain uncorrelated; correlations can be introduced only by mutual interactions.

If the system is described by a nonlinear equation, then in general the nonlinear term will produce correlations, even for subsystems, noninteracting in the sense of condition (3.1). The motion of one subsystem will be influenced by the state of the other subsystem. To illustrate this point let us consider first separately two simple physical systems each made of only one particle (particle 1 or particle 2) moving in the absence of external forces. From Eq. (2.15) we obtain the following two equations:

$$\begin{aligned} & \frac{m}{2} \frac{d^2}{dt^2} \langle \psi_k | (\mathbf{r}_k - \langle \mathbf{r}_k \rangle)^2 | \psi_k \rangle \\ &= \frac{1}{m} \left\langle \left(\frac{\hbar}{i} \nabla - \langle \mathbf{p}_k \rangle \right) \psi_k \left| \left(\frac{\hbar}{i} \nabla - \langle \mathbf{p}_k \rangle \right) \psi_k \right. \right\rangle \\ &+ 3 \langle \psi_k | [F_k(|\psi_k|^2) - G_k(|\psi_k|^2)] | \psi_k \rangle, \quad (k = 1, 2). \end{aligned} \quad (3.2)$$

Next, we consider a combined system made of both particles. Assuming that, at a given moment of time, the wavefunction ψ describing the state of this system is equal to the product of ψ_1 and ψ_2 , we obtain from Eqs. (2.15) and (3.2) the following equation for the dispersion of the first particle

$$\begin{aligned} (d^2/dt^2) \langle \psi | (\mathbf{r}_1 - \langle \mathbf{r}_1 \rangle)^2 | \psi \rangle &= (d^2/dt^2) \langle \psi_1 | (\mathbf{r}_1 - \langle \mathbf{r}_1 \rangle)^2 | \psi_1 \rangle \\ &+ (6/m) \langle \psi_1 \psi_2 | [F_{12}(|\psi_1 \psi_2|^2) - G_{12}(|\psi_1 \psi_2|^2)] \\ &- F_1(|\psi_1|^2) + G_1(|\psi_1|^2)] | \psi_1 \psi_2 \rangle. \end{aligned} \quad (3.3)$$

In particular, for the so-called nonlinear Schrödinger equation, i.e., when $F(\rho) = \lambda\rho$, the last term in Eq. (3.3) is

$$(3/m) \langle \psi_1 | [|\psi_1|^2 | \psi_1 \rangle [\lambda_{12} \langle \psi_2 | |\psi_2|^2 | \psi_2 \rangle - \lambda_1 \langle \psi_2 | \psi_2 \rangle]]. \quad (3.4)$$

Thus, the very *existence* of the second particle influences the spreading of the wave packet of the first particle. This phenomenon is a general feature of nonlinear wave mechanics.

We find the fact, that the rest of the world in the absence of forces ($U = 0$) influences the detailed behavior of an isolated particle, very unsatisfactory. No physical predictions could be made in such a theory. We shall show, however, that there exists a unique nonlinear theory in which for noninteracting subsystems no correlations are introduced by the nonlinear term. We will call this property: *the separability of noninteracting subsystems*. We mean by this that a solution of the nonlinear equation for the whole system can be constructed, as in the linear theory, by taking the product of two arbitrary solutions of the nonlinear equations for the subsystems.

Now, we will postulate the separability of noninteracting subsystems and derive the form of F . To this end let us consider a system made of two noninteracting subsystems, whose sets of coordinates will be denoted by \mathbf{r}_1 and \mathbf{r}_2 . If the wavefunctions $\psi_1(\mathbf{r}_1, t)$ and $\psi_2(\mathbf{r}_2, t)$ obey the equations ($k = 1, 2$)

$$i\hbar(\partial/\partial t) \psi_k(\mathbf{r}_k, t) = [-(\hbar^2/2m)\Delta + U(\mathbf{r}_k, t) + F_k(|\psi_k|^2)] \psi_k(\mathbf{r}_k, t), \quad (3.5)$$

then according to the separability postulate their product $\psi_1\psi_2$ has to obey the equation

$$i\hbar(\partial/\partial t)[\psi_1(\mathbf{r}_1, t) \psi_2(\mathbf{r}_2, t)] = [-(\hbar^2/2m)\Delta_1 - (\hbar^2/2m)\Delta_2 + U_1(\mathbf{r}_1, t) + U_2(\mathbf{r}_2, t) + F_{12}(|\psi_1\psi_2|^2)] \psi_1(\mathbf{r}_1, t) \psi_2(\mathbf{r}_2, t). \quad (3.6)$$

With the help of Eqs. (3.5) we obtain the following condition

$$F_1(\rho_1) + F_2(\rho_2) = F_{12}(\rho_1\rho_2). \quad (3.7)$$

The general solution of this functional equation is

$$F_1(\rho) = -b \ln(\rho a_1), \quad (3.8a)$$

$$F_2(\rho) = -b \ln(\rho a_2), \quad (3.8b)$$

$$F_{12}(\rho) = -b \ln(\rho a_1 a_2), \quad (3.8c)$$

where b is a real universal constant (having the dimension of energy), the same for all systems and their subsystems. The constants a_1 and a_2 are real and positive.

The same form of nonlinearity is obtained for spinning particles.

We conclude that the only nonlinear equation, which obeys the separability postulate, is

$$i\hbar(\partial/\partial t) \psi(\mathbf{r}, t) = [-(\hbar^2/2m)\Delta + U(\mathbf{r}, t) - b \ln(|\psi|^2 a^n)] \psi(\mathbf{r}, t), \quad (3.9)$$

where a is an arbitrary, real, and positive constant with the dimension of length and n is the dimensionality of the configuration space. The actual value of a does not have any immediate physical significance. It can even differ from one system to another. Change of a can be compensated by adding a constant to the potential energy U .

Eq. (3.9) is the only equation of type (2.1) for which every solution can have its norm changed (provided we multiply it also by a phase factor harmonically dependent on time). If $\psi(\mathbf{r}, t)$ is a solution of Eq. (3.9), then $\alpha\psi(\mathbf{r}, t) \exp[(i/\hbar) bt \ln |\alpha|^2]$ is also a solution. It is worth noting that the harmonic factor is universal

for all wavefunctions. Therefore, without any loss of generality we can use wavefunctions normalized in a certain fashion. In what follows we will assume that

$$\int d^n r \psi^*(\mathbf{r}) \psi(\mathbf{r}) = 1. \quad (3.10)$$

One more important property of Eq. (3.9) is that it allows for the separation of variables, whenever this can be done for the corresponding linear equation (e.g., the separation of the center of mass motion for an isolated system). This general property encompasses, as a special case, our separability property, which we had postulated to derive the logarithmic nonlinearity.

4. ENERGY

We have already defined the energy functional $E[\psi]$ for a general nonlinear case, up to an arbitrary additive constant. It turns out that for the logarithmic nonlinearity we can determine this constant in such a way that Planck's relation will hold for all stationary states, i.e., that

$$E[\psi] = \hbar\omega. \quad (4.1)$$

This follows directly from Eq. (2.6). For the logarithmic nonlinearity

$$F(\rho) = -b \ln(\rho a^n), \quad (4.2)$$

$$G(\rho) = -b \ln(\rho a^n) + b, \quad (4.3)$$

and if C is chosen to be $-b$, then Eq. (4.1) follows for all wavefunctions.

The logarithmic nonlinearity is the only nonlinearity for which the energy functional has such a property (see Appendix A).

With our choice of the constant C , the energy functional takes on the form

$$E[\psi] = (\hbar^2/2m) \langle \nabla \psi | \nabla \psi \rangle + \langle \psi | U | \psi \rangle - b \langle \psi | \ln(|\psi|^2 a^n) | \psi \rangle. \quad (4.4)$$

This expression clearly possesses the additivity property for noninteracting subsystems, i.e.,

$$E[\psi_1 \psi_2] = E[\psi_1] + E[\psi_2]. \quad (4.5)$$

For an isolated system the center of mass and internal degrees of freedom can be treated as two noninteracting subsystems. Therefore, it follows from (4.5) that the total energy for an isolated system, in the absence of correlations between the center of mass motion and the internal motion, is equal to the sum of the center of mass energy and the internal energy.

The additivity of the energy may also be used as a postulate to derive the logarithmic form of the nonlinearity (with the help of the method described in Appendix A). The additivity of the energy is closely related to the fact that the nonlinear part of the energy (for $b > 0$) has the form of the entropy, if ρ is interpreted as a probability distribution.

In order to find a physical interpretation of the constant b , let us consider a norm-preserving splitting of an arbitrary wavefunction for an isolated system into k nonoverlapping parts of the same form as the initial wavefunction,

$$\psi(\mathbf{r}) \rightarrow \sum_{i=1}^k (p_i)^{1/2} \psi(\mathbf{r} - \mathbf{r}_i) \quad (4.6)$$

$$\sum_{i=1}^k p_i = 1. \quad (4.7)$$

From (4.4) we obtain the following change ΔE of the total energy

$$\Delta E = -b \sum_{i=1}^k p_i \ln p_i, \quad (4.8)$$

and hence b measures the splitting energy. Therefore, if b is negative, the energy functional for isolated systems is not bounded from below. For that reason we will consider only the case $b > 0$.

For $b > 0$ the energy functional for a free particle in any number of dimensions is bounded from below. In particular, this means that the energy is bounded from below for any number of freely moving particles in three dimensions and for the center of mass motion of an isolated system.

The proof of the above assertion was given in [5], where we proved the following inequality (rewritten here with all dimensional constants inserted).

$$E[\psi] = (\hbar^2/2m) \langle \nabla \psi | \nabla \psi \rangle - b \langle \psi | \ln(|\psi|^2 a^n) | \psi \rangle \geq E_0 + (\langle \mathbf{p} \rangle^2 / 2m), \quad (4.9)$$

where

$$E_0 = nb[1 - \ln(a/l\pi^{1/2})], \quad (4.10)$$

$$l^2 = \hbar^2/2mb, \quad (4.11)$$

and $\langle \mathbf{p} \rangle$ is the average momentum. For real wave functions, (4.9) reduces to the Gross-Nelson inequality [6]. A new proof of the inequality (4.9) is given in Appendix B, where we also show that $E[\psi]$ attains its lower bound only on the Gaussian functions of the form

$$\psi(\mathbf{r}) = \exp(i\varphi) \exp((i/\hbar) \langle \mathbf{p} \rangle \cdot \mathbf{r}) G(\mathbf{r} - \boldsymbol{\eta}), \quad (4.12)$$

where

$$G(\mathbf{r}) = (l\pi^{1/2})^{-n/2} \exp(-\mathbf{r}^2/2l^2), \quad (4.13)$$

φ is an arbitrary phase and $\boldsymbol{\eta}$ is an arbitrary vector. The absolute lower bound E_0 for $E[\psi]$ is obtained when $\langle \mathbf{p} \rangle = \mathbf{0}$ and is attained only if the wavefunction ψ has the form $\exp(i\varphi) G(\mathbf{r} - \boldsymbol{\eta})$.

It should be stressed that the lower bound for $E[\psi]$ given by (4.9) is obtained for wavefunctions normalized to unity. For wavefunctions normalized to N , the right-hand side of (4.9) would read

$$N(E_0 + (\langle \mathbf{p} \rangle^2/2m) - b \ln N). \quad (4.14)$$

For N tending to ∞ , this lower bound tends to $-\infty$. Therefore, our result does not contradict the so called stability theorem proved by Derrick [7] (cf. also [8]). Of course, one must not conclude that the normalization condition will automatically lead to the existence of a lower bound of the energy functional for arbitrary nonlinearity. The simplest counterexample is obtained when $F(\rho) = -|\lambda| \rho$ (the nonlinear Schrödinger equation) in three or more dimensions. The unboundedness can be easily checked, in that case, on a family of Gaussian functions.

The energy functional in the logarithmic case without an external potential is also bounded from below for multicomponent wavefunctions, in particular this is true for particles with spin. The lower bound is the same as for one component wavefunctions,

$$E[\phi_1, \dots, \phi_r] \geq E_0, \quad (4.15)$$

and is attained only if the wave function is of the form

$$(\phi_1(\mathbf{r}), \dots, \phi_r(\mathbf{r})) = (\alpha_1, \dots, \alpha_r) G(\mathbf{r} - \boldsymbol{\eta}), \quad (4.16)$$

where $\boldsymbol{\eta}$ is an arbitrary vector and $(\alpha_1, \dots, \alpha_r)$ are constant coefficients normalized to one:

$$\sum_{i=1}^r |\alpha_i|^2 = 1.$$

The proof of this assertion is given Appendix C.

Now, we will prove a theorem about the energy functional of a free particle (particles) for a family of (normalized) wavefunctions ψ_L which are related to a certain wavefunction ψ by an arbitrary linear transformation of the \mathbf{r} -space:

$$\psi_L(\mathbf{r}) = |\det L|^{1/2} \psi(L\mathbf{r}), \quad (4.17)$$

where L is an arbitrary, real, nonsingular matrix.

The energy difference between the energies for the wavefunctions ψ and ψ_L can be written in the form

$$E[\psi] - E[\psi_L] = \text{Tr}(K) - \text{Tr}(L^T K L) + b \ln |\det L|, \quad (4.18)$$

where K is the kinetic energy matrix,

$$K_{ij} = (\hbar^2/2m) \langle \nabla_i \psi | \nabla_j \psi \rangle. \quad (4.19)$$

Let us write L as a product $L = L_0 L_1$, where L_0 is a matrix which transforms the (symmetric and positive definite) matrix $2K/b$ into the unit matrix

$$L_0^T K L_0 = (b/2)I. \quad (4.20)$$

With the help of the identity

$$\det L = [\det(LL^T)]^{1/2} \quad (4.21)$$

we can rewrite (4.18) in the form

$$\begin{aligned} E[\psi] - E[\psi_{L_0 L_1}] &= (b/2)[\text{Tr}(2K/b) - \ln \det(2K/b)] \\ &\quad - (b/2)[\text{Tr}(L_1 L_1^T) - \ln \det(L_1 L_1^T)]. \end{aligned} \quad (4.22)$$

For every symmetric, positive definite $n \times n$ matrix A the following inequality holds

$$\text{Tr} A - \ln \det A \geq n, \quad (4.23)$$

as can be seen after the diagonalization of A . Using this inequality we can draw the following conclusions from (4.22).

If the kinetic energy matrix K_{ij} is not of the form $(b/2)\delta_{ij}$, then we can always lower the energy by the transformation $L = L_0$.

If $K_{ij} = (b/2)\delta_{ij}$, then we increase the energy by the transformation $L = L_1$ unless $L_1 L_1^T = I$ (i.e., when L_1 is an orthogonal matrix).

Thus, we have proved the following theorem. For a family of wave functions related by transformations (4.17), the lowest value of the energy functional exists and is attained only on all wavefunctions for which $K_{ij} = (b/2)\delta_{ij}$.

The exact lower bound of the energy can be calculated not only for a free particle but also for the n -dimensional harmonic oscillator with the logarithmic nonlinearity. We prove in Appendix D that the following inequality holds:

$$\begin{aligned} E[\psi] &= (\hbar^2/2m) \langle \nabla \psi | \nabla \psi \rangle + (m\omega_0^2/2) \langle r^2 \rangle - b \langle \ln(|\psi|^2 a^n) \rangle \\ &\geq nb[1 - \ln(a/l(\pi)^{1/2}) - \ln(1 - \alpha^2)^{1/2}] + (1/2) n\alpha \hbar \omega_0, \end{aligned} \quad (4.24)$$

where

$$\alpha = (b/\hbar\omega_0) [(1 + (\hbar\omega_0/b)^2)^{1/2} - 1] < 1, \quad (4.25)$$

and that the lower bound is attained only if

$$\psi(\mathbf{r}) = (l(\pi(1 - \alpha^2))^{1/2})^{-n/2} \exp[-\mathbf{r}^2/2l^2(1 - \alpha^2)]. \quad (4.26)$$

We have not been able to find exact lower bounds for other potentials. However, we can prove the existence of a lower bound for almost every potential which in the linear theory leads to the Hamiltonian bounded from below. The precise condition imposed on the potential U is that the linear Hamiltonian H_λ ,

$$H_\lambda = (\mathbf{p}^2/2m) + \lambda U, \quad (4.27)$$

is bounded from below for some value of λ greater than 1. We put this condition to work by writing the energy functional as the sum of two parts:

$$\begin{aligned} E[\psi] = & \lambda^{-1}((\hbar^2/2m) \langle \nabla \psi | \nabla \psi \rangle + \lambda \langle U \rangle) \\ & + ((\lambda - 1)/\lambda) (\hbar^2/2m) \langle \nabla \psi | \nabla \psi \rangle - b \langle \ln(|\psi|^2 a^n) \rangle. \end{aligned} \quad (4.28)$$

The first part is bounded by our assumption and the second is bounded by the inequality (4.9) in which m is to be replaced by $m\lambda/(\lambda - 1)$.

5. STATIONARY STATES

We have already mentioned that wavefunctions describing stationary states are extremal points of the energy functional. It follows from here, in particular, that Gaussian wavefunctions $G(\mathbf{r} - \boldsymbol{\eta})$, for which the energy attains its absolute lower bound, must describe stationary states. The corresponding eigenvalue is equal to E_0 . These statements can be checked by a direct calculation.

We have not been able to find stationary states for a free particle other than those described by $G(\mathbf{r} - \boldsymbol{\eta})$. In the simplest, one-dimensional case we can prove that such excited states do not exist (Appendix D). In the n -dimensional case this problem remains open, but the virial theorem seems to place severe restrictions on the wavefunctions of stationary states. For the logarithmic nonlinearity the generalized virial theorem (2.16) for a free particle has the form

$$K_{ij} = \frac{1}{2} b \delta_{ij}, \quad (5.1)$$

which implies not only full "rotational symmetry" of the kinetic energy, but also the equality of kinetic energies for all stationary states. This property of stationary states follows also from the theorem about the family of ψ_L -functions which was proved in the preceding section.

The lowest energy state for a free particle in any number of dimensions has an interesting property which again singles out the logarithmic nonlinearity in a

unique fashion. This state is stable in the sense of Poincaré [9]. By this we mean that both the momentum density $m\rho\mathbf{v}$ and the stress tensor T_{ij} , calculated for this state vanish. Indeed, for the logarithmic nonlinearity we obtain from (2.11), (4.2), and (4.3)

$$T_{ij} = -\rho((\hbar^2/4m) \nabla_i \nabla_j \ln \rho + b\delta_{ij}) + m\rho v_i v_j. \quad (5.2)$$

This expression vanishes identically for $\psi = G(\mathbf{r} - \boldsymbol{\eta})$. Moreover, $G(\mathbf{r} - \boldsymbol{\eta})$ is the only wavefunction stable in the sense of Poincaré. We will prove an even stronger statement that, for a general nonlinearity described by a function $F(\rho)$, the mere existence of a wavefunction which is stable in the sense of Poincaré implies that the nonlinearity is logarithmic and the wavefunction is Gaussian.

To this end we observe that from the off-diagonal elements of the equation $T_{ij} = 0$ we obtain (for $\rho\mathbf{v} = 0$)

$$\nabla_i \nabla_j \ln \rho = 0, \quad i \neq j,$$

which implies that ρ has a product form,

$$\rho(\mathbf{r}) = \rho_1(x_1) \cdots \rho_n(x_n).$$

The diagonal elements give the following conditions

$$(\hbar^2/4m) \nabla_i \nabla_i \ln \rho_i(x_i) + G(\rho) - F(\rho) = 0.$$

The only solution of this set of conditions is

$$G(\rho) - F(\rho) = -b, \quad (5.3)$$

where b is a constant and

$$\rho(\mathbf{r}) = N \exp[-2mb(\mathbf{r} - \boldsymbol{\eta})^2/\hbar^2]. \quad (5.4)$$

The solution of (5.3) is

$$F(\rho) = -b \ln \rho + \text{const.} \quad (5.5)$$

Let us now turn to the study of stationary states in the general case, dropping the assumption that the particle is free.

Unlike in the linear theory, wavefunctions of stationary states belonging to different energies are in general not orthogonal. Instead of the orthogonality condition, we obtain now

$$(E_1 - E_2) \langle \psi_1 | \psi_2 \rangle = -b \langle \psi_1 | \ln(|\psi_1|^2/|\psi_2|^2) | \psi_2 \rangle. \quad (5.6)$$

We obtain a nontrivial condition even for $E_1 = E_2$ and the form of this condition does not depend on the value of b , as long as $b \neq 0$.

In order to complete our discussion of stationary states, we will study the influence of the logarithmic nonlinearity on several important stationary solutions of the linear theory.

As the first example we will consider a rectangular potential well with infinitely high walls, in any number of dimensions. Owing to the separability property of our nonlinear equation, we may reduce this problem to one dimension. As in the linear theory, we must find wavefunctions obeying the free equation and vanishing at the points $x = \pm d/2$. After the separation of the time dependence, the function $\psi(x)$ can be assumed to be real (see Appendix D) and one integration leads to the following ordinary differential equation for $\psi(x)$:

$$(\hbar^2/2m) (d\psi(x)/dx)^2 = [-E + b - b \ln(|\psi|^2 a)] \psi^2(x) + D, \quad (5.7)$$

where

$$D = (\hbar^2/2m)(d\psi(x)/dx)_{x=\pm d/2}^2.$$

The integration constant D must be greater than zero, since $D = 0$ leads to a trivial solution $\psi(x) = 0$. The solution of (5.7) can be expressed as the following indefinite integral

$$\int d\chi [(d\chi/d\xi)_{\xi=\pm d/2l}^2 - \chi^2 \ln \chi^2]^{-1/2} = \pm \xi, \quad (5.8)$$

where we have used dimensionless variables:

$$\xi = x/l \quad (5.9)$$

$$\chi = a^{1/2} \exp[(E/2b) - 1/2] \psi. \quad (5.10)$$

Functions $\chi(\xi)$ given implicitly by (5.8) do not satisfy, in general, the proper boundary conditions. These functions are periodic in ξ and oscillatory and the physical solutions $\chi_k(\xi)$ are those for which an integer number of half-wavelengths fits into the well, i.e.,

$$2k \int_0^{z_k} dz [z_k^2 \ln z_k^2 - z^2 \ln z^2]^{-1/2} = d/l, \quad (5.11)$$

where z_k is the maximal value of χ_k and

$$z_k^2 \ln z_k^2 = (d\chi_k/d\xi)_{\xi=\pm d/2l}^2. \quad (5.12)$$

For each physical solution the energy is to be calculated from the formula (normalization condition for ψ):

$$E_k = b \left[1 + \ln \left((l/a) \int_{-d/2l}^{d/2l} d\xi \chi_k^2(\xi) \right) \right]. \quad (5.13)$$

The second example of a stationary state in the presence of a potential is the ground state of the n -dimensional harmonic oscillator. As we already know, the wavefunction of this state is of the form (4.26) and its energy is given by the right-hand side of (4.24).

The third example is the stationary state in the Coulomb potential $U = -e^2/r$, corresponding to the ground state of the linear theory. The wavefunction of this state is of the form

$$\psi(\mathbf{r}) = C \exp(-r^2/2l - r/a_0), \quad (5.14)$$

where a_0 is the Bohr radius. The energy is given by the formula

$$E = -me^4/2\hbar^2 + 3b - b \ln(C^2 a^3). \quad (5.15)$$

The constant C is determined by the normalization condition:

$$C^{-2} = \pi l^3 [(1 + 2x^2)\pi^{1/2} \exp(x^2) \operatorname{erfc} x - 2x],$$

where $x = l/a_0$ and $\operatorname{erfc} x$ denotes the complementary error function.

6. GAUSSONS

Let us consider the time-dependent multicomponent Gaussian wavefunction $G(\mathbf{r} - \boldsymbol{\eta}) \exp(-i/\hbar E_0 t) \chi_0$, where χ_0 is a normalized, constant spinor, describing in the n -dimensional configuration space a ground state of one or several free spinning particles (or the center of mass of an isolated system). For simplicity, we will adjust the constant a in the logarithmic nonlinearity in such a way as to make E_0 equal to zero, i.e., $\ln(a/l\pi^{1/2}) = 1$.

With the help of the Galileo transformation we can boost the wavefunction $G(\mathbf{r} - \boldsymbol{\eta})$ to make it moving with the velocity \mathbf{v} in the configuration space:

$$\begin{aligned} \psi(\mathbf{r}, t) = & (l\pi^{1/2})^{-n/2} \exp(-i/\hbar)(\mathbf{p}^2/2m)t + (i/\hbar) \mathbf{p} \cdot \mathbf{r} \\ & \times \exp[-(\mathbf{r} - \boldsymbol{\eta} - \mathbf{v}t)^2/2l^2] \chi_0, \end{aligned} \quad (6.1)$$

where $\mathbf{p} = m\mathbf{v}$. It can also be checked directly, that (6.1) obeys the logarithmic wave equation.

According to formulas (4.9), (4.12), and (4.13) for each value of t the energy corresponding to the wavefunction (6.1) is the lowest possible for a particle with mean velocity \mathbf{v} .

We will call such uniformly moving Gaussian wave packets, modulated by the de Broglie plane waves—freely moving gaussons. For the sake of simplicity, only spinless gaussons will be considered.

Now, we will turn to the study of the motion of gaussons under the influence of forces. As the first example we will consider the motion under the influence of a time-dependent uniform electric field $\mathbf{E}(t)$ in three dimensions. In this case, the logarithmic wave equation possesses a family of exact solutions of the form:

$$\psi(\mathbf{r}, t) = \exp \left[(i/\hbar) \mathbf{p}(t) \cdot \mathbf{r} - (i/\hbar) \int_0^t dt' (\mathbf{p}^2(t')/2m) \right] G(\mathbf{r} - \boldsymbol{\eta}(t)), \quad (6.2)$$

where

$$\mathbf{p}(t) = e \int_0^t dt' \mathbf{E}(t') + \mathbf{p}_0, \quad (6.3)$$

and

$$\boldsymbol{\eta}(t) = (1/m) \int_0^t dt' \mathbf{p}(t') + \boldsymbol{\eta}_0. \quad (6.4)$$

Thus, in every uniform electric field the gaussons move according to classical mechanics preserving exactly their Gaussian shape.

As the second example we will study the longitudinal motion of a gausson in a uniform, time-independent magnetic field \mathbf{B} . We will choose the vector potential in the form $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$. The logarithmic wave equation,

$$i\hbar(\partial/\partial t) \psi(\mathbf{r}, t) = [-(\hbar^2/2m)\Delta + (e^2/8mc^2) (\mathbf{r} \times \mathbf{B})^2 + (e\hbar/2mci) \mathbf{B} \cdot (\mathbf{r} \times \nabla) - b \ln(|\psi|^2 a^3)] \psi(\mathbf{r}, t), \quad (6.5)$$

possesses a family of exact solutions representing gaussons moving along the magnetic field with momentum p :

$$\psi(\mathbf{r}, t) = (l\pi^{1/2})^{-1/2} \exp[-(z - \eta - (p/m)t)^2/2l^2] \times \exp[-(i/\hbar)(p^2/2m)t + (i/\hbar) pz](R\pi^{1/2})^{-1} \exp[-(x^2 + y^2)/2R^2], \quad (6.6)$$

where

$$R = l2^{1/2}((1 + (l/\lambda_0)^4)^{1/2} + 1)^{-1/2},$$

$\lambda_0 = (c\hbar/eB)^{1/2}$ is the magnetic length and the z -axis has been chosen in the direction of \mathbf{B} . The radius R of the gausson in the direction perpendicular to the magnetic field is reduced, as compared to l .

We cannot obtain exact solutions of our nonlinear equations for arbitrary electromagnetic field. However, knowing a classical trajectory $\boldsymbol{\eta}(t)$ we can associate with it a wavefunction $\psi(\mathbf{r}, t)$, which is an approximate solution of the nonlinear wave equation, provided the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$

change very little over the gaussian radius l and the magnetic length and the cyclotron radius are large as compared to l :

$$l(|\nabla E_i|/|\mathbf{E}|) \ll 1, \quad (6.7a)$$

$$l(|\nabla B_i|/|\mathbf{B}|) \ll 1, \quad (6.7b)$$

$$l(e|\mathbf{B}|/\hbar c)^{1/2} \ll 1, \quad (6.7c)$$

$$l(e|\mathbf{B}|/mc|\langle \mathbf{v}_\perp \rangle|) \ll 1. \quad (6.7d)$$

We will seek the wavefunction in question in the form

$$\psi(\mathbf{r}, t) = \exp[(i/\hbar) S(\mathbf{r}, t)] G(\mathbf{r} - \boldsymbol{\eta}(t)). \quad (6.8)$$

This wavefunction is to obey the nonlinear equation:

$$i\hbar(\partial/\partial t) \psi(\mathbf{r}, t) = [(1/2m)((\hbar/i)\nabla - (e/c)\mathbf{A})^2 + e\varphi - b \ln(|\psi|^2 a^3)] \psi(\mathbf{r}, t). \quad (6.9)$$

Since the function (6.8) is localized within the gaussian radius l in the vicinity of the trajectory $\boldsymbol{\eta}(t)$, on account of the assumptions (6.7) we can expand all slowly varying quantities in powers of $(\mathbf{r} - \boldsymbol{\eta})$ and keep only linear terms. We will choose the electromagnetic potentials in the form

$$\mathbf{A}(\mathbf{r}, t) = -(1/2) (\mathbf{r} - \boldsymbol{\eta}(t)) \times \mathbf{B}(\boldsymbol{\eta}(t), t), \quad (6.10a)$$

$$\begin{aligned} \varphi(\mathbf{r}, t) = & -(\mathbf{r} - \boldsymbol{\eta}(t)) \cdot \mathbf{E}(\boldsymbol{\eta}(t), t), \\ & - (1/2c) (\mathbf{r} - \boldsymbol{\eta}(t)) \cdot (\dot{\boldsymbol{\eta}}(t) \times \mathbf{B}(\boldsymbol{\eta}(t), t)). \end{aligned} \quad (6.10b)$$

They reproduce correctly \mathbf{E} and \mathbf{B} in the vicinity of $\boldsymbol{\eta}(t)$, if $(1/c)(\partial/\partial t) \mathbf{B} + \nabla \times \mathbf{E} = 0$.

One can check that (6.8) will be an approximate solution of (6.9) if one chooses S in the form

$$S(\mathbf{r}, t) = (m/2) \int_0^t dt' (\dot{\boldsymbol{\eta}}(t'))^2 + m(\mathbf{r} - \boldsymbol{\eta}(t)) \cdot \dot{\boldsymbol{\eta}}(t), \quad (6.11)$$

and if $\boldsymbol{\eta}(t)$ obeys the Newton equation with the Lorentz force:

$$m\ddot{\boldsymbol{\eta}} = e(\mathbf{E}(\boldsymbol{\eta}, t) + (1/c) \dot{\boldsymbol{\eta}} \times \mathbf{B}(\boldsymbol{\eta}, t)). \quad (6.12)$$

Therefore, in every electromagnetic field, sufficiently small gaussons move like classical particles.

The rest of this section will be devoted to the study of the oscillations of a single gaussian. At the beginning we will investigate certain exact solutions of the

nonlinear equation, which describe oscillations of a free gaussian. We will make the following "Ansatz" for the wavefunctions

$$\begin{aligned} \psi(\mathbf{r}, t) = & (l\pi^{1/2})^{-n/2} [\det A(t)]^{1/4} \exp[i\varphi(t)] \\ & \times \exp \left[-(1/2l^2) \sum_{i,j} x_i (A(t) + iB(t))_{ij} x_j \right], \end{aligned} \quad (6.13)$$

where $\varphi(t)$ is a real function and $A(t)$ and $B(t)$ are dimensionless, real and symmetric matrices and $A(t)$ is positive definite. From the nonlinear equation for ψ , we obtain a set of equations for A , B and φ :

$$dA/d\tau = 2(AB + BA), \quad (6.14)$$

$$dB/d\tau = 2(B^2 - A^2 + A), \quad (6.15)$$

$$d\varphi/d\tau = n - \text{Tr}(A) + (1/2) \text{Tr}(\ln A), \quad (6.16)$$

where $\tau = bt/\hbar$ is a dimensionless time parameter. The first two equations determine $A(\tau)$ and $B(\tau)$ from their initial values at $\tau = \tau_0$ and the third equation gives $\varphi(\tau)$ by integration. Eqs (6.14) and (6.15) preserve the symmetry of the matrices A and B and they also preserve the positive definiteness of A . The last property follows from the energy conservation, which can be written in terms of A and B in the form:

$$\text{Tr}(A + B^2 A^{-1} - \ln A) = \epsilon = \text{const.} \quad (6.17)$$

In the coordinate system in which A is diagonal, Eq. (6.17) reads

$$\sum_{i=1}^n \left(a_i + \sum_{j=1}^n a_i^{-1} (b_{ij})^2 - \ln a_i \right) = \epsilon, \quad (6.18)$$

and therefore $\epsilon \geq n$ and

$$\sum_{i=1}^n (a_i - \ln a_i) \leq \epsilon. \quad (6.19)$$

Thus for every i we obtain

$$0 < a_{\min}(\epsilon - n + 1) \leq a_i \leq a_{\max}(\epsilon - n + 1) < \infty, \quad (6.20)$$

where $a_{\min}(\epsilon)$ and $a_{\max}(\epsilon)$ are the solutions of the equation

$$a - \ln a = \epsilon. \quad (6.21)$$

It follows from (6.20), that not only the positive definiteness of A is preserved,

but also that the oscillations are bounded in amplitude. The dispersion of \mathbf{r} in any direction is bounded from below and from above:

$$\frac{1}{2}l^2[a_{\max}(\epsilon - n + 1)]^{-1} \leq \langle (\mathbf{n} \cdot \mathbf{r})^2 \rangle \leq \frac{1}{2}l^2[a_{\min}(\epsilon - n + 1)]^{-1}. \quad (6.22)$$

If the matrices A and B commute at $\tau = \tau_0$, then they commute at all times ($BA^{-1} - A^{-1}B$ is a constant of motion) and they can be both diagonalized by the same time-independent coordinate rotation. In this case we can give analytic solutions of Eqs. (6.14) and (6.15). To this end we write down separate equations for every eigenvalue:

$$da_i/d\tau = 4a_i b_i, \quad (6.23)$$

$$db_i/d\tau = 2b_i^2 - 2a_i^2 + 2a_i. \quad (6.24)$$

Every such pair of equations possesses the energy integral:

$$a_i + b_i^2 a_i^{-1} - \ln a_i = \epsilon_i = \text{const} \geq 1. \quad (6.25)$$

With its help we can obtain equations for a_i 's,

$$da_i/d\tau = 4a_i(a_i \ln a_i + \epsilon_i a_i - a_i^2)^{1/2}, \quad (6.26)$$

which give after integration

$$\int (da_i/a_i)(a_i \ln a_i + \epsilon_i a_i - a_i^2)^{-1/2} = 4\tau. \quad (6.27)$$

The oscillations in every eigendirection are periodic and their periods T_i are:

$$T_i = \frac{\hbar}{2b} \int_{a_{\min}(\epsilon_i)}^{a_{\max}(\epsilon_i)} (da/a)(a \ln a + \epsilon_i a - a^2)^{-1/2}. \quad (6.28)$$

Next, we will study most general oscillations of a free gaussian around its equilibrium shape, assuming however that the amplitude of these oscillations is small. The wavefunctions describing such oscillations will be written in the form (in dimensionless variables)

$$\psi(\xi, \tau) = (l\pi^{1/2})^{-n/2} \exp[-\xi^2/2 + \phi(\xi, \tau)], \quad (6.29)$$

where $|\phi|$ is assumed to be small as compared to 1. From the nonlinear wave equation we obtain (neglecting $(\nabla\phi)^2$):

$$i(\partial/\partial\tau)\phi = -\Delta\phi + 2(\xi \cdot \nabla)\phi - \phi - \phi^*. \quad (6.30)$$

The most general oscillatory solution of this equation is

$$\phi(\xi, \tau) = \sum_{p_1 \cdots p_n=0}^{\infty} c(p_1, \dots, p_n; \tau) H_{p_1}(\xi_1) \cdots H_{p_n}(\xi_n), \quad (6.31)$$

with $\sum_{i=1}^n p_i \geq 2$. The functions $H_p(\xi)$ are Hermite polynomials and the coefficients c are

$$c(p_1, \dots, p_n; \tau) = c(p_1, \dots, p_n) \times \cos[(\Sigma(\Sigma-1))^{1/2}\tau + \varphi(p_1, \dots, p_n) + i \ln(\Sigma^{1/2} + (\Sigma-1)^{1/2})], \quad (6.32)$$

where $\Sigma = p_1 + \cdots + p_n$ and $c(p_1, \dots, p_n)$ and $\varphi(p_1, \dots, p_n)$ are arbitrary real constants.

Therefore, the gaussons can undergo small oscillations with frequencies (in normal units)

$$\omega = (2b/\hbar) k(1 - 1/k)^{1/2}, \quad k = 2, 3, \dots \quad (6.33)$$

There is a $\binom{n+k-1}{n-1}$ -fold degeneracy, corresponding to each frequency. For $k = 2$ we obtain small oscillations corresponding to arbitrary oscillations discussed before (Eq. (6.13)). The nonoscillatory solutions of Eq. (6.30), corresponding to $\Sigma = 1$, describe uniform motion of the gausson (for small values of τ) and have been excluded by the assumption that the center of mass of the gausson is at rest.

We note that the assumption $|\phi| \ll 1$ does not hold for $|\xi|$ large as compared to 1, but the wavefunction is practically zero in this region.

Oscillations of a gausson can be excited by time-dependent external fields. In particular, a uniform, time-dependent magnetic field will excite modes with $k = 2$.

7. PHYSICAL INTERPRETATION

The statistical interpretation scheme of the orthodox quantum mechanics contains statements like [10]:

“Two states for which a measurement of ξ is certain to give two different results are orthogonal.”

“Any result of a measurement of a real dynamical variable is one of its eigenvalues.”

“If the measurement of the observable ξ for the system in the state corresponding to $|x\rangle$ is made a large number of times, the average of all the results obtained will be $\langle x | \xi | x \rangle$, provided $|x\rangle$ is normalized.”

These statement along with many others are based on such mathematical concepts as superposition of state vectors, completeness, and orthogonality of

eigenfunctions, representation theory, etc. All these concepts in turn, are derived from the linearity of the theory. Therefore, it may seem that a statistical interpretation of wave mechanics, in which the linear equation is replaced by a nonlinear one, will not be possible at all. In the following we will argue that it is not so and that the statistical interpretation of the nonlinear wave mechanics can be derived from the same basic postulates as in the linear theory.

First of all, let us notice that the statements like those quoted above, often used as seemingly independent postulates of the interpretation scheme, can all be derived from the (linear) Schrödinger equation and from the Born interpretation of the wavefunction. This has been recently discussed in detail by Mielnik [11].

In our nonlinear theory, based on the wave equation with the logarithmic nonlinearity (or for that matter even in a general case (2.1)), the Born postulate still can be retained, because $|\psi|^2$ obeys the continuity equation (2.9). We will adopt this postulate in its most traditional form, assuming that [12]: “ $|\psi(r, t)|^2 dV$ is the probability that the electron (regarded as a corpuscule) is in the volume element dV .” All further conclusions are to be *derived* from this postulate and from the postulated wave equation, which describes the time evolution of the system (cf. also [11]). These conclusions, in general, will be quite different for a nonlinear theory as compared to the linear one, but for a weak nonlinearity (small value of b) we would expect the differences to be numerically small in the vast majority of experimental situations.

We believe that in the interpretation of our nonlinear theory a very important role is played by the separability property and the weak superposition principle. These two properties make it possible to introduce wavefunctions to describe states of an isolated system, when correlations with the rest of the world are present. As we have already said in Section 3, the separability guarantees that no correlations between the subsystem and the rest of the world are introduced by the nonlinearity in the wave equation. If such correlations have been introduced by the interaction in the past, then the wavefunction will have the form

$$\psi(x, X, t) = \sum_i \phi_i(x, t) \Phi_i(X, t), \quad (7.1)$$

where $\phi_i(x, t)$ are normalized functions of the subsystem variables. For nonoverlapping ϕ_i 's each term in this sum will evolve independently and will keep its product form. If we are interested in the subsystem alone, we can describe its state, as a mixed state, by the set of wavefunctions $\phi_i(x, t)$ with probability weights $\|\Phi_i\|$. It should be noted that the choice of normalization of the functions ϕ_i is not essential, since a change of normalization affects only the oscillating phase factor.

Two properties discussed above enable us also to describe the measurement in the nonlinear theory along similar lines as in the linear theory. Due to the separa-

bility, before the interaction we can describe the state of the whole system, made of the object and the apparatus, by a product wavefunction. The necessary condition for a measurement is that after the interaction the wavefunction of the whole system will have to separate into several parts, each part corresponding to a different "position of the pointer." The weak superposition principle guarantees that these parts, as not overlapping, evolve independently. Each part is a product wavefunction describing a macroscopically distinguishable state of the pointer and a certain state of the rest of the system. The separability guarantees that after the interaction the factorization will be preserved in time. Thus, after the measurement the state behaves like a mixed state.

We hope that the foregoing discussion has shown that the interpretation of the measurement process along the same lines as in the linear theory would not have been possible for nonlinearities different than the logarithmic nonlinearity.

The most obvious, and at same time a very important, difference between the linear and our nonlinear wave mechanics is the existence of stable wave products—the gaussons. Since the radius of the gausson $l = \hbar/(2mb)^{1/2}$ decreases with the increase of its mass, gaussons representing the center of mass motion of large (isolated) systems will be small even for small b . This property enables us to separate microobjects from macroobjects, according to whether the extension of an object is much smaller or much larger than the size of its center of mass gausson. In particular, for a spherical object of radius R and mass density d , the center of mass gausson has the radius

$$l = \hbar((8\pi/3) bd)^{-1/2} R^{-3/2}. \quad (7.2)$$

The critical size is obtained when $l = R = R_0$, where

$$R_0 = \hbar^{2/5}((8\pi/3) bd)^{-1/5}. \quad (7.3)$$

All objects with $R \ll R_0$ are microobjects and those with $R \gg R_0$ are macroobjects. For microobjects, wave properties are, in general, seen. For macroobjects, wave properties will not show up in their center of mass motion. This follows from our discussion of the motion of small gaussons in external fields. It should be observed that even though the splitting energy of a gausson $b \ln 2$ is mass independent, such a splitting would require forces acting on the center of mass coordinates to vary significantly over the dimensions of the gausson. Forces of this type are not available for macroobjects, whose gausson radii are much smaller than interatomic distances.

For objects whose sizes are of the order of R_0 we introduce the name mesoobjects. One can expect that mesoobjects have properties which are qualitatively different from those of microobjects and macroobjects. It is worth noting that the value

of R_0 , which determines the position of mesoworld on the scale of linear dimensions changes very little over a wide range of values of b and d .

If we want to apply our nonlinear theory to the description of the physical reality we must first of all make sure that all predictions of this theory will not run into contradiction with known experimental facts. This clearly means that our constant b *must be very small*, because otherwise microobjects would behave like classical particles. The question immediately arises how big b could be on the basis of the present knowledge of the microworld.

We have to admit that at present there is no experimental evidence of any nonlinear term in the Schrödinger equation. Therefore, all we can do now is to set an upper limit on b from precision measurements in atomic physics. We think that the most restrictive result is obtained from the Lamb shift measurement in hydrogen for 2S–2P transitions. The agreement between theory and experiment is here at the level of 1×10^5 Hz. Since the logarithmic nonlinearity would contribute corrections to this shift of the order of b , we obtain as an upper limit for b :

$$b < 4 \times 10^{-10} \text{ eV.} \quad (7.4)$$

This restriction on b leads to the following inequalities for R_0 and for the gaussian radius of electrons

$$R_0 > 18d^{-1/5} \text{ \AA}, \quad (7.5)$$

where d is in g cm^{-3} and

$$l_{el} > 10 \text{ \mu m.} \quad (7.6)$$

As we have already mentioned there are no experimental facts which would lead to a lower limit for b . However, we are tempted to speculate that mesoworld may have something to do with the world of large organic molecules. In other words we expect R_0 to be of the order of the linear dimensions of such molecules (say, smaller than the dimensions of the smallest virus). In this way we would obtain the upper bound for R_0

$$R_0 < 50 \text{ \AA}, \quad (7.7)$$

which is equivalent (for $d = 1 \text{ g cm}^{-3}$) to

$$b > 2.5 \times 10^{-12} \text{ eV}, \quad (7.8)$$

$$l_{el} < 120 \text{ \mu m.} \quad (7.9)$$

We find it very interesting that the two bounds for b , obtained from totally different arguments, are consistent and that the predicted values of b range only over two orders of magnitude.

APPENDIX A

We will prove here that if the condition

$$\langle [G(|\psi|^2) - F(|\psi|^2)] \rangle + C = 0, \quad (\text{A.1})$$

where C is a constant, is valid for all stationary states, then the nonlinearity must be logarithmic. The proof will be constructed in two steps.

In the first step we will investigate whether, for a given, normalized wavefunction $\psi(\mathbf{r})$, there exists a potential $U(\mathbf{r})$ and an eigenvalue $\hbar\omega$ such that $\psi(\mathbf{r})$ describes a stationary state, i.e., such that

$$[-(\hbar^2/2m) \Delta + U(\mathbf{r}) + F(|\psi|^2)] \psi(\mathbf{r}) = \hbar\omega\psi(\mathbf{r}). \quad (\text{A.2})$$

We will show that the necessary and sufficient condition for the existence of such $U(\mathbf{r})$ and $\hbar\omega$ is the vanishing of the divergence of the current. By multiplying (A.2) by $\psi^*(\mathbf{r})$ and then subtracting or adding the corresponding complex conjugate equation, we obtain

$$\nabla \cdot (\psi^* \nabla \psi - \nabla \psi^* \cdot \psi) = 0, \quad (\text{A.3})$$

$$U(\mathbf{r}) = \hbar\omega - F(|\psi|^2) + (\hbar^2/4m |\psi|^2) (\Delta |\psi|^2 - 2\nabla \psi^* \cdot \nabla \psi). \quad (\text{A.4})$$

On the other hand, the potential of the form (A.4) with the divergence condition (A.3) guarantees the validity of (A.2). It follows, therefore, that the condition (A.1) must hold for practically all normalized wavefunctions.

In the second step, we will choose $\psi(\mathbf{r})$ to be equal to a constant $\lambda^{1/2}$ inside some region of volume λ^{-1} and equal to zero outside this region. Such function can be considered an approximation to a certain smooth function. Now, (A.1) reduces to

$$\lambda^{-1}[\lambda(G(\lambda) - F(\lambda))] + C = 0. \quad (\text{A.5})$$

With the help of the definition of G , we obtain the following differential equation for F .

$$\lambda(dF/d\lambda) = C. \quad (\text{A.6})$$

The general solution is

$$F(\lambda) = C \ln(\lambda a^n), \quad (\text{A.7})$$

where a is an arbitrary constant.

APPENDIX B

The energy functional for a free particle in n dimensions can be written in the form

$$E[\psi] = (l/a)^n b \epsilon[\Phi], \quad (\text{B.1})$$

where

$$\begin{aligned} \epsilon[\Phi] &= \int d^n \xi [\nabla \Phi^* \cdot \nabla \Phi - |\Phi|^2 \ln |\Phi|^2], \\ \xi &= \mathbf{r}/l \quad \text{and} \quad \Phi = a^{n/2} \psi. \end{aligned} \quad (\text{B.2})$$

We will prove that for all functions Φ normalized to unity and such that $\langle \xi \rangle = 0$ the following inequality holds

$$\epsilon[\Phi] \geq n(1 + \ln \pi^{1/2}) \quad (\text{B.3})$$

and that the lower bound is attained only on the Gaussian function of the form (apart from a trivial, overall phase factor)

$$\Phi(\xi) = \pi^{-n/4} \exp(-\xi^2/2). \quad (\text{B.4})$$

Let us define a real, positive function $\xi(f)$,

$$\xi(f) = (1/\pi^{1/2}) \left[(n/2) \Gamma(n/2) \int d^n \xi \theta(|\Phi| - f) \right]^{1/n}, \quad (\text{B.5})$$

where θ is the step function

$$\begin{aligned} \theta(x) &= 0 & x \leq 0, \\ &= 1 & x > 0. \end{aligned}$$

The function $\xi(f)$ is right continuous and strictly decreasing to zero in the interval $[0, |\Phi|_{\max}]$. The inverse function $f(\xi)$ exists and is defined over the range of the function $\xi(f)$. In this range, $f(\xi)$ is strictly decreasing from its maximal value $M = |\Phi|_{\max}$ to zero. We can extend this function to all values of its argument ξ in a unique way by the requirements of continuity and monotonicity. In this way we can associate with every wavefunction Φ a real, monotonic and spherically symmetric function $f(|\xi|)$, which satisfies the assumptions of the theorem. The average value of ξ is zero by spherical symmetry and the norm of f is equal to 1 by the following, more general equality:

$$\begin{aligned} \int d^n \xi F(|\Phi|^2) &= - \int_0^\infty df F(f^2) (d/df) \int d^n \xi \theta(|\Phi| - f) \\ &= - \pi^{n/2} [(n/2) \Gamma(n/2)]^{-1} \int_0^\infty df F(f^2) (d/df) \xi^n(f) \\ &= \int d^n \xi F(f^2(\xi)). \end{aligned} \quad (\text{B.6})$$

Let us first consider only such functions Φ for which $f(\xi)$ is strictly decreasing, for all values of ξ , from 0 to ∞ . For those functions we define the following averaging operation.

$$\{g\}_f = (1/w(f)) \int d^n \xi g(\xi) \delta(|\Phi| - f), \quad (\text{B.7})$$

where

$$w(f) = \int d^n \xi \delta(|\Phi| - f). \quad (\text{B.8})$$

We will show that the kinetic energy corresponding to our new function $f(\xi)$ does not exceed the kinetic energy for the initial wavefunction Φ .

The area A_f of the surface $|\Phi| = f$ can be represented in the form

$$A_f = w(f) \{|\nabla|\Phi|\}_f. \quad (\text{B.9})$$

This surface (not necessarily singly connected) encloses the volume V_f

$$V_f = [(n/2) \Gamma(n/2)]^{-1} (\xi(f) \pi^{1/2})^n. \quad (\text{B.10})$$

Since the smallest surface enclosing a given volume is spherical, the following inequality holds

$$w(f) \{|\nabla|\Phi|\}_f \geq 2\pi^{n/2} \Gamma(n/2)^{-1} (\xi(f))^{n-1}. \quad (\text{B.11})$$

This inequality becomes the equality for every value of f only when all surfaces $|\Phi| = f$ are singly connected and spherical (spheres corresponding to smaller values of f contain spheres corresponding to larger values). In what follows we will also make use of the two inequalities:

$$\{(\nabla|\Phi|)^2\}_f \geq \{|\nabla|\Phi|\}_f^2, \quad (\text{B.12})$$

and

$$|\nabla\Phi|^2 \geq (\nabla|\Phi|)^2. \quad (\text{B.13})$$

The first (Schwartz) inequality becomes an equality only when $|\nabla|\Phi||$ is constant over the whole surface $|\Phi| = f$. The second inequality becomes an equality only for functions Φ whose phase is constant in every singly connected region of nonvanishing $|\Phi|$.

The kinetic energy for $f(\xi)$ can be written in the form

$$\begin{aligned} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty d\xi \xi^{n-1} \left(\frac{df}{d\xi}\right)^2 &= \frac{2\pi^{n/2}n}{\Gamma(n/2)} \int_M^0 df \xi^{2n-2}(f) \left[\frac{d\xi^n(f)}{df}\right]^{-1} \\ &= \frac{4\pi^n}{(\Gamma(n/2))^2} \int_0^M df \frac{\xi^{2n-2}(f)}{w(f)}. \end{aligned} \quad (\text{B.14})$$

Using this equality and the inequalities (B.11)–(B.13) we obtain finally for the kinetic energy

$$\begin{aligned} \int d^n \xi (\nabla f)^2 &\leq \int df w(f) \{ |\nabla | \Phi | | \}_f^2 \\ &\leq \int_0^M df w(f) \{ (\nabla | \Phi |)^2 \}_f = \int d^n \xi (\nabla | \Phi |)^2 \\ &\leq \int d^n \xi \nabla \Phi^* \cdot \nabla \Phi. \end{aligned} \quad (\text{B.15})$$

Thus, it follows from (B.6) and (B.15) that for functions Φ such that f is strictly decreasing for all values of ξ

$$\epsilon[\Phi] \geq \epsilon[f]. \quad (\text{B.16})$$

The equality holds only when Φ coincides with its function f (apart from a constant phase factor).

All functions Φ such that f is not strictly decreasing can be approximated with arbitrary accuracy by functions considered before. Therefore, also for these functions the inequality (B.16) holds.

Now, we will use for function f the inequality

$$-\langle \ln \rho \rangle - \langle \ln \tilde{\rho} \rangle \geq n(1 + \ln \pi) \quad (\text{B.17})$$

proved by us in [5]. In the present case

$$\begin{aligned} \rho(\xi) &= f^2(\xi), \\ \tilde{\rho}(\kappa) &= \tilde{f}^2(\kappa), \\ \tilde{f}(\kappa) &= (2\pi)^{-n/2} \int d^n \xi \exp(i\kappa \cdot \xi) f(\xi). \end{aligned}$$

The Fourier transform \tilde{f} is, of course, spherically symmetric and real. From (B.17) we obtain

$$\epsilon[f] \geq n(1 + \ln \pi) + \int d^n \kappa \tilde{\rho}[\kappa^2 + \ln \tilde{\rho}]. \quad (\text{B.18})$$

The last integral is a strictly concave functional of $\tilde{\rho}$ and attains its lower bound $-n \ln \pi^{1/2}$ only on the Gaussian function (cf. [5])

$$\tilde{\rho}(\kappa) = \pi^{-n/2} \exp(-\kappa^2). \quad (\text{B.19})$$

We obtain finally

$$\epsilon[f] \geq n(1 + \ln \pi^{1/2}), \quad (\text{B.20})$$

and the equality holds only when $\tilde{\rho}$ is equal to (B.19) and hence when

$$f(\xi) = \pi^{-n/4} \exp(-\xi^2/2). \quad (\text{B.21})$$

Our proposition (B.3) follows directly from (B.16) and (B.21). Since the Gaussian function is strictly decreasing for all ξ , the equality in (B.3) holds only for the Gaussian function (B.4), apart from a constant phase factor.

The restriction $\langle \xi \rangle = 0$ can be easily removed by a translation of the origin of the coordinate system. In this way we obtain that for all normalized functions Φ , the inequality (B.3) holds and that the lower bound is attained only on the Gaussian functions of the form

$$\Phi(\xi) = \pi^{-n/4} \exp[-(\xi - \xi_0)^2/2]. \quad (\text{B.22})$$

The following, more general, theorem is also valid. For normalized function Φ such that

$$-i \int d^n \xi \Phi^* \nabla \Phi = \kappa_0, \quad (\text{B.23})$$

the inequality

$$\epsilon[\Phi] \geq \kappa_0^2 + n(1 + \ln \pi^{1/2}) \quad (\text{B.24})$$

holds and the lower bound is attained only on the modulated Gaussian function of the form

$$\Phi(\xi) = \pi^{-n/4} \exp(i\kappa_0 \cdot \xi) \exp[-(\xi - \xi_0)^2/2]. \quad (\text{B.25})$$

The proof of (B.24) can be obtained from the previous theorem by the substitution

$$\Phi = \exp(i\kappa_0 \cdot \xi) \Phi_0, \quad (\text{B.26})$$

where Φ_0 obeys the restriction

$$-i \int d^n \xi \Phi_0^* \nabla \Phi_0 = 0. \quad (\text{B.27})$$

APPENDIX C

The existence of the lower bound (4.15) can be proved with the help of the elementary logarithmic inequality:

$$-\sum_{i=1}^r |\alpha_i|^2 \rho_i \ln \left(\sum_{j=1}^r |\alpha_j|^2 \rho_j \right) \geq -\sum_{i=1}^r |\alpha_i|^2 \rho_i \ln \rho_i, \quad (\text{C.1})$$

which is valid for nonnegative ρ_i 's and for α_i 's fulfilling the normalization condition. The following chain of inequalities is obtained for the energy

$$\begin{aligned}
 E[\alpha_1\tilde{\phi}_1, \dots, \alpha_r\tilde{\phi}_r] &= \sum_{i=1}^r |\alpha_i|^2 (\hbar^2/2m) \langle \nabla\tilde{\phi}_i | \nabla\tilde{\phi}_i \rangle \\
 &\quad - b \int d^n r \left(\sum_{i=1}^r |\alpha_i|^2 |\tilde{\phi}_i|^2 \right) \ln \left(\sum_{k=1}^r |\alpha_k|^2 \ln(|\tilde{\phi}_k|^2 a^n) \right) \\
 &\geq \sum_{i=1}^r |\alpha_i|^2 E[\tilde{\phi}_i] \geq E_0,
 \end{aligned} \tag{C.2}$$

where wavefunctions $\tilde{\phi}_i$ are all normalized to unity. The last inequality becomes the equality only when all $\tilde{\phi}_i$'s (corresponding to nonvanishing α_i 's are of the form $\exp(i\varphi_i) G(\mathbf{r} - \boldsymbol{\eta}_i)$ with arbitrary $\boldsymbol{\eta}_i$'s and φ_i 's. The first inequality in (C.2) becomes the equality only when moduli of $\tilde{\phi}_i$'s corresponding to nonvanishing α_i 's are all equal everywhere. Therefore, the functional $E[\alpha_1\tilde{\phi}_1, \dots, \alpha_r\tilde{\phi}_r]$ attains its lower bound only on wavefunctions of the form (4.15).

The first inequality in (C.2) also can be used to prove the following statement. The energy $E = E[\psi_R + i\psi_I]$ for a complex wavefunction cannot be smaller than the lower of the two energies $E_R = E[\langle \psi_R | \psi_R \rangle^{-1/2} \psi_R]$ and $E_I = E[\langle \psi_I | \psi_I \rangle^{-1/2} \psi_I]$. In addition, the equality $E = \min(E_R, E_I)$ not only implies, as in the linear theory, that $E = E_R = E_I$, but also that $\psi_R^2(r)$ is proportional to $\psi_I^2(r)$.

APPENDIX D

The nonlinear one-dimensional wave equation for stationary states of a free particle reads

$$-(d^2/d\xi^2)\chi - (1 + \ln |\chi|^2)\chi = 0, \tag{D.1}$$

where dimensionless variables χ and ξ are defined by (5.9) and (5.10).

Let us consider an interval (ξ_1, ξ_2) of the ξ axis such that ξ does not vanish inside this interval but vanishes at both ends (ξ_1 and ξ_2 may be equal to $+\infty$ and $-\infty$). The continuity equation in one dimension requires that, for square-integrable wavefunctions of stationary states, the current vanishes. Hence, without any loss of generality, we can assume that χ is real and positive in the interval (ξ_1, ξ_2) . Under this assumption, we multiply (D.1) by $d\chi/d\xi$ and integrate with respect to ξ :

$$(d\chi/d\xi)^2 = -\chi^2 \ln \chi^2 + D. \tag{D.2}$$

Let us suppose that the integration constant D is negative. In that case, χ can not

vanish for finite ξ , so that $\xi_1 = \infty$ and $\xi_2 = -\infty$. However, it follows from (D.2) that for $D \neq 0$, χ can not vanish together with its derivative at infinity. If for positive D , one or both points ξ_1 and ξ_2 lie at infinity the function χ will again not vanish with its derivative at infinity. On the other hand, if for positive D both points ξ_1 and ξ_2 are finite, the derivative of χ does not vanish at these points. Then, the continuity of the first derivative forces χ to be periodic (cf. our discussion of solutions in the potential well in Section 5) and thus not square integrable in the whole ξ space.

We conclude, that $D = 0$ and (D.2) reduces in the interval (ξ_1, ξ_2) to

$$(d\chi/d\xi)^2 = -2\chi^2 \ln \chi. \quad (\text{D.3})$$

The integration of this equation gives

$$\chi = \exp[-(\xi - \xi_0)^2/2]. \quad (\text{D.4})$$

This function does not vanish for finite ξ 's and hence ξ_1 and ξ_2 are infinite. The additional solution $\chi = 1$ of (D.3) is not a solution of the original equation (D.1).

We have shown, therefore, that the only stationary states in one dimension are of the Gaussian form.

APPENDIX E

The energy functional for the nonlinear harmonic oscillator has the form

$$E[\psi] = (\hbar^2/2m)\langle \nabla\psi | \nabla\psi \rangle + (m\omega_0^2/2)\langle \mathbf{r}^2 \rangle - b\langle \ln(|\psi|^2 a^n) \rangle. \quad (\text{E.1})$$

It is convenient to divide $E[\psi]$ into two parts:

$$E[\psi] = (1 - \alpha^2) [(\hbar^2/2m)\langle \nabla\psi | \nabla\psi \rangle - (b/(1 - \alpha^2))\langle \ln(|\psi|^2 a^n) \rangle] + \alpha^2(\hbar^2/2m)\langle \nabla\psi | \nabla\psi \rangle + (m\omega_0^2/2)\langle \mathbf{r}^2 \rangle, \quad (\text{E.2})$$

where α is given by (4.25). The first part of (E.2) is bounded from below, on account of our inequality (4.9), by the expression

$$nb[1 - \ln(a/l\pi^{1/2}) - \ln(1 - \alpha^2)^{1/2}], \quad (\text{E.3})$$

and attains its lower bound only for Gaussian functions of the form

$$\psi(\mathbf{r}) = (l(\pi(1 - \alpha^2))^{1/2})^{-n/2} \exp[-(\mathbf{r} - \boldsymbol{\eta})^2/2l^2(1 - \alpha^2)]. \quad (\text{E.4})$$

The second part can be shown, with the help of the Heisenberg uncertainty relation, to be not smaller than

$$\frac{n^2 \alpha^2 \hbar^2}{8m} \langle \mathbf{r}^2 \rangle^{-1} + \frac{m \omega_0^2}{2} \langle \mathbf{r}^2 \rangle, \quad (\text{E.5})$$

and is equal to (E.5) only for Gaussian functions centered at $\mathbf{r} = 0$. In turn, (E.5) has its lower bound equal to $\frac{1}{2} n \alpha \hbar \omega_0$ and attains this bound only if $\langle \mathbf{r}^2 \rangle = n \alpha \hbar / 2m \omega_0$.

Putting all these facts together, we obtain

$$E[\psi] \geq nb[1 - \ln(a/l\pi^{1/2}) - \ln(1 - \alpha^2)^{1/2}] + (1/2) n \alpha \hbar \omega_0. \quad (\text{E.6})$$

The equality holds only for ψ given by (E.4) with $\eta = 0$.

Note added in proof. The procedure used by us in Appendix B to lower the kinetic energy is known in the context of linear wave mechanics as the spherical decreasing rearrangement (V. Glaser, A. Martin, H. Grosse and W. Thirring in "Essays in Honor of V. Bargmann" (B. Simon and A. Wightman, Ed.), Princeton Univ. Press, 1976).

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