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Nonlinear Structure of the Electromagnetic Vacuum

Iwo Bialynicki-Birula

Institute for Theoretical Physics, Polish Academy of Sciences, Lotnikow 32/46, 02-668 Warsaw, Poland

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Abstract

In this paper I review those properties of quantum electrodynamics which lead to nonlinear effects in the dynamics of the electromagnetic field.

1. Introduction

One of the most profound changes in our understanding of Nature brought about by quantum field theory is the realization that the vacuum is not empty. The vacuum state – defined as the ground state of the system – is alive with the oscillations of all the virtual particles of the Universe. The fact that the vacuum is populated with virtual particles has been recognized already by Dirac, who in his famous paper on the quantum theory of radiation [1] described it in the following way:

"The light-quantum has the peculiarity that it apparently ceases to exist when it is in one of its stationary states, namely, the zero state, in which its momentum and therefore also its energy, are zero. When a light-quantum is absorbed it can be considered to jump into this zero state, and when one is emitted it can be considered to jump from the zero state to one in which it is physically in evidence, so it appears to have been created. Since there is no limit to the number of light-quanta that may be created in this way, we must suppose that there are an infinite number of light-quanta in the zero state."

During the sixty years that have passed since the discovery of the non-empty vacuum, a lot has been learned about its properties. Gradually it became clear that the complete description of the quantum field can be extracted from the properties of its vacuum. When it is left undisturbed, the system in its ground state does not show any signs of life; the vacuum state remains the vacuum state. Every perturbation, however, produces excitations of the system. The full dynamics of the field is described by the response of the system in its vacuum state to different external perturbations. This information is stored in the vacuum to vacuum transition amplitude treated as a functional of the fields describing the perturbations. The logarithm of this amplitude is called the effective action. In the classical limit, the effective action becomes just the ordinary action functional of the classical field. Thus, the effective action contains on one hand all the information about the quantum system and on the other hand it allows an almost classical interpretation.

The exact evaluation of the effective action is tantamount to a complete solution of the problem and, therefore, can only be carried out for free fields. In quantum electrodynamics and in other theories with a weak coupling we can approximately evaluate the effective action by perturbation theory.

The aim of this lecture is to describe in some detail the applications of the effective action theory in quantum electrodynamics to a restricted class of problems. I shall only deal with external perturbations due to slowly changing electromagnetic fields. Even in this narrow domain one finds many interesting and novel physical phenomena.

2. Heisenberg–Euler–Weisskopf–Schwinger effective Lagrangian

The history of the effective action is just over fifty years old. It begun in 1936, when Werner Heisenberg and his two students Euler and Kockel decided to study the scattering of light on light according to the Dirac hole theory. As a result of this study, they have not only wrote down the photon–photon cross-section, but they have also been able to determine the Lagrangian for a slowly changing electromagnetic field in interaction with the virtual pairs of electrons and positrons. The paper [2] in which this was presented is a true masterpiece. It contains a sophisticated, gauge covariant method of handling the infinities in quantum field theory (a simple version of what 30 years later became the Wilson expansion method), including rudimentary renormalization, superb classical mathematical analysis combined with deep physical insights.

I would like to recall very briefly the procedure used by Heisenberg and Euler in their derivation of the effective action. They studied the behavior of the second-quantized electron field in the presence of a constant electromagnetic field. Due to the interaction of the Dirac-sea electrons with the electromagnetic field, the energy of the system will differ from its free-field value even in the ground state of the electron fields, i.e., when no real electrons or positrons are present. The starting point of their calculation of the effective action was the following formula for the energy density of the system

$$U = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \sum_i \psi_i^* \left[c\boldsymbol{\alpha} \cdot \left(\frac{\hbar}{i} \nabla + \frac{e}{c} \boldsymbol{A} \right) + \beta mc^2 \right] \psi_i - U_0, \quad (1)$$

where the summation extends over all negative energy states and U_0 is the value of the energy (the vacuum energy) in the absence of the external field. The above formula is obtained by taking the expectation value of the energy operator of the electron field in the ground state of the system. The sum over all occupied states in this formula is highly singular. Heisenberg and Euler had to use a sophisticated point splitting method to obtain the correct (gauge invariant) result. This gauge invarint point splitting method has been reinvented in the fifties without a proper credit being given to its originators.

The evaluation of the sum over the negative energy states can be attempted when we have a complete set of the wave functions at our disposal. Such is the case for homogeneous and time independent electric and magnetic fields. It is sufficient to consider only the case of parallel $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, since every $(\boldsymbol{E}, \boldsymbol{B})$-pair becomes parallel after an appropriate Lorentz transformation and the Dirac equation is relativistically invariant. For homogeneous fields there is an additional simplification: we only need the spectrum of the Hamiltonian, because the energy density does not depend on the point. The Lagrangian density for the constant electromagnetic field was obtained from the energy density (1) with the use of the Legendre transformation formula.

$$L = \boldsymbol{E}\cdot\frac{\partial U}{\partial \boldsymbol{E}} - U. \tag{2}$$

Due to the relativistic invariance, the Lagrangian can only depend on the two scalar expressions S and P that can be formed from the components of the field tensor $F_{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu}$

$$\begin{aligned} S &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2),\\ P &= -\tfrac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \boldsymbol{E}\cdot\boldsymbol{B}. \end{aligned} \tag{3}$$

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\lambda\varrho}F_{\lambda\varrho}. \tag{4}$$

After the charge renormalization, the Lagrangian of Heisenberg and Euler can be written in the form,

$$L_{\mathrm{HE}} = S - \frac{\alpha}{8\pi^2}\int_0^\infty \mathrm{d}\eta\,\frac{\mathrm{e}^{-\eta}}{\eta}\left[P\,\frac{\operatorname{Re}\cosh(\eta X/F_c)}{\operatorname{Im}\cosh(\eta X/F_c)} - F_c^2/\eta^2 + \frac{2}{3}S\right], \tag{5}$$

where the critical or the characteristic field strength F_c is constructed from elementary constants as follows

$$F_c = \frac{m^2c^3}{e\hbar} \tag{6}$$

and X is a complex combination of the two invariants,

$$X^2 = -2S + 2\mathrm{i}P. \tag{7}$$

The expression (5) has been rederived in a slightly different manner by Weisskopf [3]. Schwinger [4] confirmed this result using his elegant and much simpler proper time technique in quantum field theory. In the coordinate system in which the $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors are parallel, the ratio of the real to the imaginary part of the hyperbolic cosine factorizes into a product,

$$P\,\frac{\operatorname{Re}\cosh(\eta X/F_c)}{\operatorname{Im}\cosh(\eta X/F_c)} = \frac{B_0}{\tanh(\eta B_0/F_c)}\,\frac{E_0}{\tan(\eta E_0/F_c)}. \tag{8}$$

The field magnitudes B_0 and E_0 are the lengths of the field vectors in the chosen coordinate system. They are (apart from the factors ± 1 and $\pm\mathrm{i}$) the eigenvalues of the field tensor $F_{\mu\nu}$ and are given as the following algebraic functions of the original invariants of the field tensor,

$$B_0 = \operatorname{Re}(-2S + 2\mathrm{i}P)^{1/2} = ((S^2+P^2)^{1/2} - S)^{1/2}, \tag{9a}$$

$$E_0 = \operatorname{Im}(-2S + 2\mathrm{i}P)^{1/2} = ((S^2+P^2)^{1/2} + S)^{1/2}. \tag{9b}$$

The Heisenberg–Euler Lagrangian can be expanded into an asymptotic series by simply expanding the integrand in eq. (5) and integrating term by term. This task is greatly simplified if one uses a computer and a language (like for example MuSimp) that allows for symbolic manipulations of algebraic expressions. To exhibit the asymptotic character of this expansion, I shall write down here the first 5 terms of this expansion ($m = 1 = c$)

$$\begin{aligned} L_{\mathrm{HE}} \approx{}& S + \frac{2\alpha^2}{45}(4S^2 + 7P^2) + \frac{32\pi\alpha^3}{315}(8S^3 + 13SP^2)\\ &+ \frac{256\pi^2\alpha^4}{945}(48S^4 + 88S^2P^2 + 19P^4) + \frac{4096\pi^3\alpha^5}{1485}\\ &\times(160S^5 + 332S^3P^2 + 127SP^4) + \frac{131072\pi^4\alpha^6}{225225}\\ &\times(44224S^6 + 102736S^4P^2 + 55270S^2P^4\\ &+ 4359P^6) + \cdots \end{aligned} \tag{10}$$

The asymptotic expansion entirely misses the exponentially small part of the Lagrangian (5); the Taylor expansion of the integrand around zero "does not see" the poles at all multiples of π exhibited by the formula (8). These poles make the integral meaningless unless we use a prescription how to avoid these poles when we evaluate the integral. If we want a real Lagrangian we must use the principal value prescription. The effective action, however, can be and in general is complex. Moreover, its imaginary part has a clear physical interpretation, since it gives the probability p_v for the vacuum to remain the vacuum,

$$p_v = |\mathrm{e}^{\mathrm{i}W}|^2 = \mathrm{e}^{-2\operatorname{Im}W}. \tag{11}$$

The pair creation by an external field is (like the photon creation by an external current) governed by the Poisson distribution, since all pairs are created independently. Therefore, the expression $2\operatorname{Im}W$ is the probability of the creation of a single pair. For the constant electromagnetic field this probability is infinite because of the infinite extension of the field in space and in time. The imaginary part of the effective Lagrangian, however, would be finite and would describe the probability density of pair creation: the probability per unit volume and per unit time. As has been shown by Schwinger [4], the iε-prescription of quantum field theory is properly implemented by replacing the principal value prescription in the evaluation of (5) by the integration contour lying above the real axis. With the use of the well known identity

$$\lim_{\varepsilon\to 0}\frac{1}{x - \mathrm{i}\varepsilon} = \mathrm{PV}\,\frac{1}{x} + \mathrm{i}\pi\delta(x). \tag{12}$$

we can then explicitly evaluate the imaginary part of the lagrangian as a sum of the contributions from all the poles of the integrand

$$2\operatorname{Im}L = \frac{\alpha E_0 B_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{\mathrm{e}^{-\pi n F_c/E_0}}{\tanh(\pi n B_0/E_0)}. \tag{13}$$

In what follows, I shall be interested only in the real part of the Lagrangian which governs the nonlinear dynamics of the electromagnetic field. A very detailed discussion of the pair creation by intense fields can be found in recent papers by Ritus [5] and Nikishov [6].

The asymptotic expansion does not tell us anything about the behavior of the effective action for large values of the fields, which must be obtained from the integral inself. For large values of B_0 the asymptotic behavior of the Heisenberg–

Euler Lagrangian is given by the formula [5]

$$L_{HE} \approx \frac{\alpha B_0^2}{6\pi} (\ln (B_0/\gamma\pi F_c) + 6\zeta'(2)/\pi^2), \tag{14}$$

where γ is the Euler constant ($\ln \gamma = 0.57722 \ldots$) and $\zeta(x)$ is the Riemann zeta function ($6\zeta'(2)/\pi^2 = -0.56996 \ldots$). For large values of E_0, the asymptotic behaviour can be obtained form (14) by the replacement $B_0 \to -iE_0$.

3. General properties of nonlinear electrodynamics

The effective Lagrangian of quantum electrodynamics given by the Heisenberg–Euler formula has been obtained under the simplifying assumption that the electromagnetic field is constant. The validity of this expression, however, extends beyond this limit. One can show by studying the contributions to this expression obtained from quantum field theory that eq. (5) will be a good approximation to the effective action even for fields that vary in space and in time, provided this variation is sufficiently slow. The scale is set by the Compton wave of the electron λ_C and by the time τ_C associated with this length,

$$\begin{aligned} \lambda_c &= \hbar/mc = 386.16\,\text{fm}, \\ \tau_C &= \hbar/mc^2 = 1.288 \times 10^{-21}\,\text{s}. \end{aligned} \tag{15}$$

Hence, the dimensionless small parameters will be λ/λ_C and τ/τ_C, where λ and τ characterize the space and the time variation of the electromagnetic field. What makes this approximation even better is that owing to the relativistic and to the gauge invariance the corrections to the effective Lagrangian (5) must start not with one but with two derivatives of the field $F_{\mu\nu}$. The lowest corrections, therefore, are quadratic in the dimensionless parameters defined above. The dynamics of all macroscopic electromagnetic fields (including even the optical region) is very accurately described by this effective Lagrangian and it is worthwhile to study the properties of this object.

Nonlinear theory of the electromagnetic field has an interesting history predating the arrival of quantum electrodynamics. Such a theory has been for the first time considered by Mie [7], who attempted to construct a field-theoretic model of the classical electron. His nonlinear theory, however, lacked gauge invariance and for that reason was found unacceptable. In 1933 Born, joined afterwards by Infeld, have revived Mie's theory in a different, gauge invariant form. In a series of papers [8–11] they developed a general theory of nonlinear electrodynamics and also produced an exceptionally interesting model of such a theory.

The Lagrangian of nonlinear electrodynamics of Born and Infeld and the effective Lagrangian of Heisenberg and Euler are two special examples of the Lagrangians of the electromagnetic field that depend only on the two relativistic invariants S and P of the field tensor. Theories based on such Lagrangians share several interesting general properties that can be described without specifying the explicit form of the Lagrange function.

Every theory of such a type is described by the well known set of Maxwell equations (I consider only the source-free case),

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{16a}$$

$$\partial_t \boldsymbol{D} = \nabla \times \boldsymbol{H}, \qquad \nabla \cdot \boldsymbol{D} = 0, \tag{16b}$$

supplemented by the (nonlinear) constitutive relations,

$$\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{D}, \boldsymbol{B}), \quad \boldsymbol{H} = \boldsymbol{H}(\boldsymbol{D}, \boldsymbol{B}). \tag{17}$$

The choice of the fields $\boldsymbol{D}$ and $\boldsymbol{B}$ in the constitutive relations as the primary variables and of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ as functions of those variables is suggested by the form of the Maxwell equations when they are compared with the canonical equations of classical dynamics. The assumption, made by Born and Infeld, that $\boldsymbol{D}$ and $\boldsymbol{B}$ form the canonical pair of variables, leads to a consistent canonical formulation of the nonlinear theory.

In the relativistic tensor notation, the Maxwell equations have the form

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0, \tag{18a}$$

$$\partial_\mu H^{\mu\nu} = 0, \tag{18b}$$

where

$$H^{\mu\nu} = -\frac{\partial L}{\partial F_{\mu\nu}} = \frac{\partial L}{\partial S} F^{\mu\nu} + \frac{\partial L}{\partial P} \tilde{F}^{\mu\nu}. \tag{19}$$

The energy, the momentum and the angular momentum of the nonlinear electromagnetic field are built from the components of the symmetric energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu} = F^{\mu}_{\lambda} H^{\lambda\nu} - g^{\mu\nu} L. \tag{20}$$

Its components

$$T^{00} = \boldsymbol{E} \cdot \boldsymbol{D} - L, \quad T^{0i} = (\boldsymbol{E} \times \boldsymbol{H})^i = T^{i0} = (\boldsymbol{D} \times \boldsymbol{B})^i, \tag{21a}$$

$$T^{ij} = -E^i D^j - H^i B^j + \delta^{ij}(L + \boldsymbol{H} \cdot \boldsymbol{B}), \tag{21b}$$

are the energy density, the energy flux, the momentum density and the Maxwell stress tensor, respectively. It follows from the Maxwell equations that the energy-momentum tensor obeys the continuity equation and as a result of this the energy, the momentum and the angular momentum are conserved.

The coordinate system in which the field vectors are parallel, used in the previous section, on the basis of the formulas (21a) may indeed be called the "rest frame of the electromagnetic field". In this frame both the energy flux and the momentum density vanish. One can show from the transformation formulas for the field vectors that in order to get to the rest frame from a given coordinate system, one must move in the direction of the Poynting vector $\boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{D} \times \boldsymbol{B}$ with the velocity v given by the formula

$$v/c = \chi - \sqrt{\chi^2 - 1}, \tag{22}$$

where

$$\chi = \frac{\boldsymbol{E}^2 + \boldsymbol{B}^2}{2|\boldsymbol{E} \times \boldsymbol{B}|}. \tag{23}$$

The total energy E of the field,

$$E = \int d^3 r\, T^{00}, \tag{24}$$

when expressed in terms of the canonical variables $\boldsymbol{B}$ and $\boldsymbol{D}$, becomes the Hamiltonian of the system.

$$H = \int d^3 r\, \boldsymbol{E}(\boldsymbol{D}, \boldsymbol{B}) \cdot \boldsymbol{D} - L(\boldsymbol{E}(\boldsymbol{D}, \boldsymbol{B}), \boldsymbol{B}). \tag{25}$$

The energy of the field obtained from the Heisenberg–Euler Lagrangian exhibits a pathological property: it is not bounded from below. This can be seen, for example, from the asymptotic formula (14), since for a pure magnetic field the Hamiltonian (25) is equal to the minus Lagrangian. However, this "breakdown of the vacuum" takes place for field strengths that exceed all reasonable limits of applicability of quantum electrodynamics, namely when

$$\ln (B_0/F_c) \simeq 1/\alpha. \tag{26}$$

A review of the canonical formalism and of the Legendre transformations for a general nonlinear theory of the electromagnetic field may be found in my earlier article [12] and in our monograph [13]. Here I would like to concentrate on the concrete physical electromagnetic processes in the vacuum.

4. Propagation of photons in the background field

In a nonlinear field theory the superposition principle does not hold or figuratively speaking any electromagnetic field feels the presence of other fields in the region where it exists. The photons in the vacuum feel the presence of other photons and, of course, they also feel the presence of macroscopic fields made of many photons. Since the nonlinear effects become more pronounced when the field strength is comparable with the characteristic value F_c, we expect to see the nonlinear effects more easily in the strong field regime. More easily does not mean easily; the nonlinearities of the electromagnetic field are extremely weak. There are two reasons for this weakness. The first reason is that the lowest order correction to the Lagrangian, which is linear in the fine structure constant α, is unobservable: it is absorbed by the renormalization of the field strength. The second reason is the exceedingly slow (logarithmic) change of the nonlinear corrections with the field strength as compared to the Maxwell Lagrangian. As a result of all this, the nonlinear term does not amount to more than 1%, even for fields that are million times stronger than F_c. Still the mere presence of nonlinear effects makes such a great conceptual difference that their study is of interest. This is especially true in such a refined theory as quantum electrodynamics, where even minute departures from the linear behavior may have observable consequences. Some of them, like for example the effects of the vacuum polarization by the Coulomb field of the nucleus on the atomic energy levels, have already been seen.

In this section I would like to discuss the effects of strong fields on the propagation of photons. In the next section I will discuss the processes that involve a change in the number of photons, as for example the photon splitting or the creation and annihilation of photons.

Since a single photon represents a weak excitation of the electromagnetic field, its propagation can be described in the linear approximation. The presence of a strong background field however, due to the nonlinearities of the field equations, will influence the photon propagation. Instead of the standard Maxwell equations obeyed by the photon wave function, we must consider the linearized version of the nonlinear Maxwell equations (16) and (17). This is most easily obtained by expanding the electromagnetic field in the nonlinear Lagrangian around its strong field value, say $F_{\mu\nu}$ and keeping only the quadratic terms of this expansion (the linear terms vanish if the strong field is an exact solution of the field equations). I shall follow here our earlier papers [12, 14]. The quadratic part in the weak field $f_{\mu\nu}$ reads

$$L^{(2)} = \gamma_S(-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}) + \gamma_P(-\tfrac{1}{4}f_{\mu\nu}\tilde{f}^{\mu\nu}) + \tfrac{1}{8}\gamma_{SS}(F_{\mu\nu}f^{\mu\nu})^2 + \tfrac{1}{8}\gamma_{PP}(\tilde{F}_{\mu\nu}f^{\mu\nu})^2 + \tfrac{1}{4}\gamma_{SP}F_{\mu\nu}f^{\mu\nu}\tilde{F}_{\lambda\varrho}f^{\lambda\varrho}, \tag{27}$$

where the coefficients γ represent the first and the second derivatives of the Lagrangian with respect to S and P evaluated at the strong field values of these invariants. In what follows, I shall assume that the strong field is constant. In this case, the field equations for the photon field $f_{\mu\nu}$ can be easily solved since they represent linear equations with constant coefficients. The plane wave solution with positive energy will be expressed in terms of the vector potential A_μ,

$$A_\mu = \varepsilon_\mu(k)\,\mathrm{e}^{-ik\cdot x}. \tag{28}$$

The components of the polarization vector obey the following set of linear algebraic equations resulting from the Lagrangian (27)

$$(k^\mu k_\nu + \alpha_{SS}a^\mu a_\nu + \alpha_{PP}\tilde{a}^\mu\tilde{a}_\nu + \alpha_{SP}(\tilde{a}^\mu a_\nu + a^\mu\tilde{a}_\nu))\varepsilon^\nu = k^2\varepsilon^\mu, \tag{29}$$

where

$$\alpha_{SS} = \gamma_{SS}/\gamma_S, \quad \alpha_{PP} = \gamma_{PP}/\gamma_S, \quad \alpha_{SP} = \gamma_{SP}/\gamma_S, \tag{30}$$

$$a^\mu = F^{\mu\nu}k_\nu, \quad \tilde{a}^\mu = \tilde{F}^{\mu\nu}k_\nu. \tag{31}$$

The dispersion law, i.e., the relation between the frequency ω and the wave vector $\boldsymbol{k}$, is obtained as the solubility condition of the set of eq. (29), from the vanishing of the determinant. The calculations are simplified by the observation that the polarization vector can be written as a sum of the vectors a_μ, $\tilde{a}_\mu$, and k_μ — the only vectors appearing in eqs. (29). Moreover, the part parallel to k_μ separates from the rest (the result of the gauge invariance). In this manner the problem becomes effectively only two-dimensional and its solution can be written in the form

$$k^2 = a^2\lambda_\pm, \tag{32}$$

where

$$\lambda_\pm = \tfrac{1}{2}(\alpha_{SS} + \alpha_{PP} - 2S\Gamma \pm \Delta^{1/2})(1 - P^2\Gamma - 2(S\alpha_{PP} - P\alpha_{SP}))^{-1}, \tag{33}$$

$$\Delta = (\alpha_{SS} - \alpha_{PP} - 2S\Gamma)^2 + (2\alpha_{SP} - 2P\Gamma)^2, \tag{34}$$

$$\Gamma = \alpha_{SS}\alpha_{PP} - \alpha_{SP}^2. \tag{35}$$

Equation (32) is a homogeneous quadratic equation in the components of the four-vector k_μ. The coefficient $\lambda_\pm$ may be called the birefringence index, because its two values determine two forms of the dispersion law: for two different polarizations.

It is worth mentioning at this point that the only two theories that do not exhibit birefringence are the linear Maxwell theory and the Born–Infeld nonlinear electrodynamics. For other nonlinear theories the velocity of the photon propagation in the presence of an electromagnetic field will depend in general on its polarization. The information about the velocity can be obtained from the dispersion law

$$\omega = [(\boldsymbol{k}\cdot\mathscr{S})^2 + \boldsymbol{k}^2 - (\boldsymbol{k}\times\mathscr{E})^2 - (\boldsymbol{k}\times\mathscr{B})^2]^{1/2} + \boldsymbol{k}\cdot\mathscr{S}, \tag{36}$$

where

$$\mathscr{E} = E/(\lambda_\pm^{-1} + E^2)^{1/2}, \quad \mathscr{B} = B/(\lambda_\pm^{-1} + E^2)^{1/2},$$

$$\mathscr{S} = \mathscr{E} \times \mathscr{B}. \tag{37}$$

The presence of the $(\boldsymbol{k} \cdot \mathscr{S})$ term in the dispersion law leads to the existence of a directional effect: the speed of light changes when the direction of its propagation is reversed. This is an analog of the Fresnel–Fizeau effect in moving media. The external electromagnetic field in the vacuum is very much like the moving medium, as long as it has a non-vanishing Poynting vector. This directional effect is not present only in those coordinate systems in which the Poynting vector vanishes. This should explain the term "rest frame of the electromagnetic field" used by me before. The difference δv between the speed of light in the direction of $\mathscr{S}$ and in the opposite direction is given by twice the length of $\mathscr{S}$ (in units of c).

In the lowest order, we obtain from eq. (36) the following expressions for the group velocity of light $\boldsymbol{v}_\pm$

$$\boldsymbol{v}_\pm/c = \boldsymbol{n}[1 - \tfrac{1}{2}\kappa_\pm(\boldsymbol{E}^2 + \boldsymbol{B}^2 + (\boldsymbol{n}\cdot\boldsymbol{E})^2 + (\boldsymbol{n}\cdot\boldsymbol{B})^2)] + \kappa_\pm[\boldsymbol{E}(\boldsymbol{n}\cdot\boldsymbol{E}) + \boldsymbol{B}(\boldsymbol{n}\cdot\boldsymbol{B}) + \boldsymbol{E}\times\boldsymbol{B}], \tag{38}$$

where $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{k}$ and the coefficients $\kappa_\pm$ have the values

$$\kappa_+ = 4\alpha/45\pi F_c^2, \quad \kappa_- = 7\alpha/45\pi F_c^2. \tag{39}$$

The last term in eq. (38) is responsible for the Fresnel–Fizeau effect, since it does not contain the vector $\boldsymbol{n}$.

For laboratory fields the changes in the speed of light are very small, of the order of 10^{-30}c, but the effect clearly indicates that the QED vacuum acts as a medium.

5. Transmutations of photons in the presence of electromagnetic fields

In the previous section I have discussed the effects of the electromagnetic field on the propagation of the photon. An even more striking phenomenon, however, is its decay in a constant field into several less energetic photons moving in the same direction [14, 15].

As I have said in the Introduction, the original aim of Heisenberg and his students was to describe the photon–photon scattering via the exchange of virtual electron–positron pairs. The quartic term in the effective Lagrangian (10) describes this scattering in the low frequency limit. The role of one of the photons, however, may be played by the external electromagnetic field. If one of the initial photons is replaced by the external field, we obtain the splitting of a photon into two photons moving in the same direction. Such a process can not take place in the absence of the external field, even though it is allowed by the energy and momentum conservation, because the available phase-space is zero. This phase-space is proportional to k^2 and it is the modification (32) of the dispersion law that allows for the photon splitting. Again, the effect is extremely small. The 50 keV photon will travel through the magnetic field of 10^8 T (found near a neutron star) for 10^{14} m before it splits into two photons. Splitting into more than two photons is also possible, but it is suppressed by higher powers of k^2 in the multiphoton phase-space.

Constant external field can only supply momentum and angular momentum to allow for photon disintegration, but can not supply the energy. In space and time dependent external fields new phenomena occur. In order to see how much fields influence the photon dynamics we can again consider the Lagrangian (27) quadratic in the photon field. The field equations for $f_{\mu\nu}$ will still be linear but the coefficients wil now be functions of space and time. Positive energy solutions of such equations will mix with the negative energy solutions — photons are created from the vacuum by the external field and annihilated into the vacuum. These phenomena are basically the same as those seen in various nonlinear devices in the laboratories (for example, four-wave mixing), but their intensity is so low that it would be extremely hard to see them.

One more interesting feature of the nonlinear photon generation by the external field is that these photons are produced in their squeezed states [16]. This allows one in principle to tell that they have been produced by a nonlinear mechanism as opposed to the generation by the sources.

In this review I have described the vacuum of quantum electrodynamics as a nonlinear medium capable of exhibiting many exciting effects. It is a great pity that these effects are so incredibly weak. If the fine structure constant were much larger, the theory of the electromagnetic field would have been much more interesting!

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