SOLUTIONS OF VLASOV–MAXWELL EQUATIONS FOR A MAGNETICALLY CONFINED RELATIVISTIC COLD PLASMA

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Special representation of the distribution function is employed to obtain new solutions of the coupled Vlasov–Maxwell equations. This approach combines two modes of description used in plasma physics: magnetohydrodynamics and the theory of orbits. Using this method, we described plasma configurations confined in one and two directions in space, with plane and cylindrical symmetry, respectively.

1. Introduction

The aim of this paper is to describe some new self-consistent solutions of the relativistic Vlasov–Maxwell equations. These solutions describe stationary streams of electrons moving against the neutralizing ionic background. The motion of the electrons is confined by their own static magnetic field.

Several exact solutions of the Vlasov–Maxwell equations for neutralized cold plasma beams have been reported in the past1–5). Our solutions, in the simplest case of plane symmetry, are closely related to those obtained recently by Peter, Ron and Rostoker5). These solutions are obtained with the use of a fairly general method which perhaps may be also applied to plasma configurations different from the ones treated in this paper.

2. General outline of the method

We will seek solutions of the coupled set of Vlasov–Maxwell equations for the distribution functions \( f_\alpha(r, p, t) \) describing various plasma components and

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the electric and magnetic fields self-consistently modified by the plasma particles,

\[
\left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + e_a (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{p}} \right] f_a = 0 ,
\]

\[\partial_t \mathbf{E} = -\nabla \times \mathbf{B} + 4\pi \mathbf{j} , \quad \nabla \cdot \mathbf{E} = 4\pi \rho , \tag{1b}\]

\[\partial_t \mathbf{B} = \nabla \times \mathbf{E} , \quad \nabla \cdot \mathbf{B} = 0 . \tag{1c}\]

The sources of the electromagnetic field, \(\rho\) and \(\mathbf{j}\), are determined by the distribution functions

\[\rho(r, t) = \sum_a e_a \int d^3p f_a(r, p, t) , \tag{2a}\]

\[j(r, t) = \sum_a e_a \int d^3p v f_a(r, p, t) . \tag{2b}\]

For relativistic particles the relation between the velocity \(v\) and the kinetic momentum \(p\) has the form

\[v - \frac{p}{\sqrt{m_a^2 + p^2}} (c - 1) . \tag{3}\]

We will restrict ourselves to special solutions of eqs. (1) for which the motion of each plasma component at every point decomposes into a discrete number of streams,

\[f_a(r, p, t) = \sum_A n_A^a(r, t) \delta(p - \pi^a_A(r, t)) . \tag{4}\]

Different streams are labelled by the superscript \(A\), whereas the subscript \(a\) always refers to different plasma components. The number of different streams does not have to be specified at this point, but it will be set equal to two or four for the electron component of the plasma in our specific examples.

As was to be expected, upon substituting our Ansatz (4) into Vlasov equation (1a), we obtain for each stream the equations of relativistic hydrodynamics of a charged fluid without the pressure term:
\[ \partial_t n^A_a + \nabla \cdot (n^A_a v^A_a) = 0 , \quad (5a) \]
\[ \partial_t \pi^A_a + (v^A_a \cdot \nabla) \pi^A_a = e_a (E + v^A_a \times B) , \quad (5b) \]

where
\[ v^A_a = \frac{\pi^A_a(r,t)}{\sqrt{m^2_a + (\pi^A_a(r,t))^2}} . \quad (6) \]

The charge and current densities, which enter the Maxwell equations, are given as sums over all streams and over all plasma components:

\[ \rho(r,t) = \sum_{A,a} e_a n^A_a(r,t) , \quad (7a) \]
\[ j(r,t) = \sum_{A,a} e_a n^A_a(r,t)v^A_a(r,t) . \quad (7b) \]

In the simplest case, when there is only one stream for each plasma component, our Ansatz reduces to the known hydrodynamic representation of the distribution function for cold plasma\(^6\). Our generalization of this known Ansatz to two (or more) \( \delta \)-functions for one plasma component enables us to find new types of solutions. This approach combines two modes of description used in plasma physics: the magnetohydrodynamics and the theory of orbits. While locally the dynamics of the system is completely determined by partial differential equations of many-fluid hydrodynamics coupled to the Maxwell equations, global properties of solutions will be inferred from the particle picture. We will visualize the streams of electrons at each space point as resulting from the continuous motion of electrons bending in their own magnetic field. In this way we can obtain solutions confined in a finite region of space.

The set of eqs. (5)–(7), together with the Maxwell equations, will be solved in sections 3 and 4 under the assumptions of special symmetry. For symmetric plasma configurations our method of solution bears a strong resemblance to the \( \delta \)-function method introduced by Longmire\(^1\) and Mjolsness\(^2\) and used subsequently by various authors\(^3\)–\(^5\).

Two cases of symmetry will be treated in our paper: plane symmetry and cylindrical symmetry. In both cases we shall discuss only time-independent solutions, although by Lorentz transforming our solutions to a moving frame, one can obtain also time-dependent solutions.
3. Solutions with plane symmetry

Solutions considered in this section are characterized by the magnetic field pointing in the z direction and depending on the x variable only,

$$B = (0, 0, B(x)).$$

For simplicity, we shall assume that the ions are at rest and distributed in such a way that at each point their charge neutralizes the charge of electrons. For the electron distribution function we shall assume at first the sum of two δ-functions (two streams) in eq. (4). The coupled hydrodynamic-Maxwell equations reduce in this case to the following set (we omit the index α and $A = 1, 2$):

$$\nabla \cdot (n^A v^A) = 0, \quad (9a)$$

$$(v^A \cdot \nabla) \pi^A = e(v^A \times B), \quad (9b)$$

$$\nabla \times B = 4\pi e(n^1 v^1 + n^2 v^2). \quad (9c)$$

If it were not for the presence of two streams (two terms on the r.h.s. of eq. (9c)), we would not obtain any new, interesting solutions.

Under the assumption of plane symmetry, the density fields $n^A$ and the velocity fields $v^A$ depend only on the variable $x$ and eqs. (9) reduce to

$$\frac{d}{dx} (n^A v^A_x) = 0, \quad (10a)$$

$$v^A_x \frac{d}{dx} \pi^A_x = e v^A_y B, \quad (10b)$$

$$v^A_x \frac{d}{dx} \pi^A_y = -e v^A_x B, \quad (10c)$$

$$v^A_x \frac{d}{dx} \pi^A_z = 0, \quad (10d)$$

$$\frac{dB}{dx} = -4\pi e(n^1 v^1_y + n^2 v^2_y), \quad (10e)$$

$$n^1 v^1_x + n^2 v^2_x = 0, \quad (10f)$$
\[ n^1 v^1_2 + n^2 v^2_2 = 0. \quad (10g) \]

General solutions of these equations do not satisfy the global constraints of "stream continuity" discussed in section 2. However, under a special choice of the integration constants, the two streams of electrons form a matching pair, leading to global solutions. In order to see how this comes about, we shall rewrite the set of equations (10) in terms of a different set of variables. To this end, we observe first that the densities of both streams can be eliminated by integrating eq. (10a) and using the constraint equation (10f):

\[ n^1 = \frac{K}{v^1_1}, \quad n^2 = -\frac{K}{v^2_2}, \quad (11) \]

where \( K \) is an integration constant. For definiteness, we shall assume that the first stream is the one which has a nonnegative component of its velocity in the \( x \) direction; therefore \( K > 0 \). In order to satisfy eq. (10d) and eq. (10g) with two streams only, we must set \( v^1_1 = 0 = v^2_2 \). Eqs. (10b) and (10c) guarantee conservation of kinetic energy in the magnetic field. Hence, we can parametrize the velocity components in the following way:

\[ (v^A_x(x), v^A_y(x)) = v^A(x) \cos \psi_A(x), \sin \psi_A(x)), \quad (12) \]

where the \( v^A \)'s do not depend on \( x \).

The two angles \( \psi_A \) and the magnetic field \( B \) obey the following set of coupled equations:

\[ \frac{d}{dx} \sin \psi_A = -\frac{eB}{p_A}, \quad (13a) \]

\[ \frac{dB}{dx} = -4\pi eK (\tan \psi_1 - \tan \psi_2). \quad (13b) \]

where

\[ p_A = \frac{mv_A}{\sqrt{1 - v^2_A}}. \quad (14) \]

Since the first and the second stream have been assumed to be a continuation of each other, the moduli of momenta must be equal,

\[ p_1 = p_2 = p. \quad (15) \]
However, this does not imply that the angles are equal. On the contrary, in order to obtain a nontrivial solution we must keep them different and the global continuity requirement gives

\[ \psi_2 = \psi_1 + \pi. \]  

(16)

This finally leads to the set of two equations:

\[ \frac{d \sin \psi}{d \xi} = -\beta, \]  

(17a)

\[ \frac{d \beta}{d \xi} = -\tan \psi, \]  

(17b)

where \( \psi = \psi_1 \) and we have introduced the dimensionless variables

\[ \dot{\xi} = x \sqrt{\frac{8\pi e^2 K}{\rho}}, \]  

(18a)

\[ \beta = \frac{B}{\sqrt{8\pi K p}}. \]  

(18b)

Eqs. (17) can be reduced to a quadrature, because they possess one integral,

\[ \frac{1}{2} \beta^2 + \cos \psi = a, \]  

(19)

where \( a \) is a constant. This integral has a simple physical interpretation. It is the \( xx \) component of the stress tensor and its independence of \( x \) expresses the vanishing of the divergence of the stress tensor. Since by our convention \( \cos \psi \) is always positive, so is the constant \( a \).

With the use of (19), the solution of eqs. (17) can be written as

\[ \pm(\xi_0 - \xi) = \frac{1}{\sqrt{2}} \int_{\phi_0}^{\phi} \frac{d\psi \cos \psi}{\sqrt{a - \cos \psi}}, \]  

(20a)

\[ \beta = \pm \sqrt{2(a - \cos \psi)}, \]  

(20b)

where \( \xi_0 \) is the (trivial) integration constant, which only fixes the origin of the \( \xi \) coordinate. Both choices of the sign in eqs. (20) give a solution.
Depending on whether the constant $a$ is greater or less than one, the solution has a different character. In the first case, the strong field regime, the angle $\psi$ varies continuously over the whole range,

$$-\pi/2 \leq \psi \leq \pi/2,$$

whereas in the second case, the weak field regime, its range is restricted by either of the two conditions

$$-\pi/2 \leq \psi \leq -\arccos a \quad \text{or} \quad \arccos a \leq \psi \leq \pi/2.$$

These two types of solutions are shown in figs. 1 and 2. As seen in these figures, all orbits lie inside a strip of width $d$. Thus, our solutions describe plasma configurations confined to a slab of the width $d$. The absolute value of the

Fig. 1. Plane symmetry, strong field regime ($\beta_0 = -1.0040$). (a) Magnetic field $\beta$ and velocity angle $\psi$ as functions of coordinate $\xi$. (b) Individual particle trajectory.
outside magnetic field $B_0$ is always the same on both sides, but for $a < 1$, the field changes its sign. Since at the boundary $\cos \psi = 0$, the constant $a$ is related to $B_0$ by the formula

$$a = \left( \frac{B_0}{B_{cr}} \right)^2, \quad (23)$$

where

$$B_{cr} = \sqrt{16\pi Kp} \quad . \quad (24)$$
From (20a) one can determine the dependence of \( d \) on \( B_0 \):

\[
d = \frac{1}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \frac{d\psi \cos \psi}{\sqrt{(B_0/B_{cr})^2 - \cos \psi}} , \quad |B_0| > B_{cr} .
\] (25a)

\[
d = \frac{1}{\sqrt{2}} \int \frac{d\psi \cos \psi}{\sqrt{(B_0/B_{cr})^2 - \cos \psi}} , \quad |B_0| < B_{cr} .
\] (25b)

The dependence of \( d \) on \( \xi \) is shown in fig. 3.

The integral (20a) can also be explicitly evaluated and expressed in terms of elliptic integrals. Upon the substitutions

\[
\cos \phi_1 = \sin(\psi/2) , \quad |B_0| > B_{cr} .
\] (26a)

\[
\cos^2 \phi_2 = 1 - k^2 \cos^2(\psi/2) , \quad |B_0| < B_{cr} ,
\] (26b)

where

\[
k^2 = \frac{2}{1 + (B_0/B_{cr})^2} .
\] (27)

this integral reduces to the canonical, Legendre forms of the elliptic integrals. In this way, we obtain

\[
\pm \xi = \left( \frac{2}{k} - k \right) F(\phi_1|k) - \frac{2}{k} E(\phi_1|k) + \text{const} . , \quad |B_0| > B_{cr} ,
\] (28a)

\[
\pm \xi = F(\phi_2|k^{-1}) - 2E(\phi_2|k^{-1}) + \text{const} . , \quad |B_0| < B_{cr} ,
\] (28b)

where \( F \) and \( E \) are the elliptic integrals of the first and the second kind.

In the limiting case, when \( |B_0| = B_{cr} \), both expressions (28) are valid and the elliptic integrals reduce to the elementary functions

\[
\pm \xi = \ln|\tan(\psi/4)| + 2\cos(\psi/2) + \text{const} .
\] (29)

This special solution has been known for a long time. It describes a neutralized plasma beam (in the Rosenbluth limit) which originates at infinity and is deflected by a magnetic barrier. The case \( |B_0| > B_{cr} \) has been recently studied.
in detail by Peter, Ron and Rostoker\cite{5} in their analysis of plasma injection into a magnetic field. Since we are interested here in plasma configurations which are confined in the $x$ direction, our choice of boundary conditions is different and, therefore, our solutions differ slightly from those of ref. 5. Moreover, our choice of boundary conditions allows for the existence of solutions also when $|B_0| < B_0$.

In the simple case of two electron streams we were forced to assume that each electron orbit lies in the plane. In order to allow for motion in the direction of the magnetic field, we must introduce a second pair of streams. In this case the equations (10a)--(10d) have, of course, the same form, but now $A = 1, 2, 3, 4$, whereas eqs. (10e)--(10g) contain contributions from all four streams:

$$\frac{dB}{dx} = -4\pi e(n^1v^1_x + n^2v^2_x + n^3v^3_x + n^4v^4_x), \quad (30a)$$
\[ n^1 v_x^1 + n^2 v_x^2 + n^3 v_x^3 + n^4 v_x^4 = 0, \quad (30b) \]
\[ n^1 v_z^1 + n^2 v_z^2 + n^3 v_z^3 + n^4 v_z^4 = 0. \quad (30c) \]

We shall again eliminate the densities \( n^A \) by integrating the continuity equations (10a). In order to satisfy eqs. (30b) and (30c) we choose the following integration constants:

\[ n^1 v_x^1 = K, \quad n^2 v_x^2 = -K, \quad n^3 v_x^3 = K, \quad n^4 v_x^4 = -K, \quad (31) \]

and the following relations between the \( z \) components:

\[ v_x^1 = v_x^2 = v_x^3 = v_x^4 = v_x. \quad (32) \]

Thus, the net current has only the \( y \) component. In view of eq. (10d), the velocity component \( v_z \) in the field direction must be constant. Global continuity requirements for electron orbits are satisfied if we choose

\[ v_x^1 = v_x^3 = v_x = v_x \cos \psi(x), \quad v_x^2 = v_x^4 = -v_x, \quad (33) \]
\[ v_y^1 = v_y^2 = v_y^3 = v_y^4 = v_y = v_y \sin \psi(x), \quad (34) \]

where \( v_x \) is the magnitude of the projection of the electron velocity on the \( xy \) plane. In dimensionless units, the equations for the velocity angle \( \psi \) and the magnetic field \( \beta \) are exactly the same as for two streams. Therefore, the magnetic field is unchanged. The trajectories now have a spiral form, but their projections on the \( xy \) plane are the same as before.

Under the special choice of velocities of our four streams (31)-(34), the distribution function can be reduced to a single product of \( \delta \)-functions

\[ f(r, p, t) = 2K \pi_z \delta(E(p) - H)\delta(p_y - \pi_y)\delta(p_z^2 - \pi_z^2), \quad (35) \]

where

\[ E(p) = \sqrt{m^2 + p^2}, \quad (36) \]

and

\[ H(r) = \sqrt{m^2 + (\pi(r))^2}. \quad (37) \]

For \( \pi_z = 0 \), this expression coincided with the formula for the distribution function used in ref. 5.
Plane-symmetric solutions are confined only in one direction. To achieve confinement in two directions, we will consider in the next section the case of cylindrical symmetry.

4. Solutions with cylindrical symmetry

We shall assume the same general form (4) of the electron distribution function as before and start from eqs. (9) for the stationary configuration. These equations will now be solved under the assumption of cylindrical symmetry. The components of all physical quantities in cylindrical coordinates $r, \theta$ and $z$ will depend on $r$ only and the magnetic field will be taken again in the $z$ direction. For simplicity, we restrict ourselves to two streams, which excludes any motion in the direction of the magnetic field, $v_z^A = 0$. Generalization to four streams to allow for motions in the $z$ direction proceeds in exactly the same way as in the case of plane symmetry discussed in the previous section. Under these assumptions, eqs. (9) reduce to

\[
\frac{d}{dr} (r n^A v^A_r) = 0, \quad (38a)
\]

\[
v_r^A \frac{d}{dr} \pi^A_r - \frac{1}{r} v^A_\theta \pi^A_\theta = -ev^A_z B, \quad (38b)
\]

\[
v_r^A \frac{d}{dr} \pi^A_\theta + \frac{1}{r} v^A_r \pi^A_\theta = ev^A_r B, \quad (38c)
\]

\[
\frac{dB}{dr} = -4\pi e (n^1 v^1_\theta + n^2 v^2_\theta), \quad (38d)
\]

\[
n^1 v^1_r + n^2 v^2_r = 0. \quad (38e)
\]

As before, we use the continuity equations (38a) and eq. (38e) to eliminate $n^A$,

\[
n^1 = \frac{\Gamma}{r v^1_r}, \quad n^2 = -\frac{\Gamma}{r v^2_r}, \quad (39)
\]

where again $\Gamma$ is a positive integration constant.

Assumptions concerning the existence of global solutions, analogous to those
made in the preceding section, lead to the following set of equations for dimensionless variables:

\[
\frac{d \sin \psi}{d \rho} = -\frac{\sin \psi}{\rho} - \beta, \quad (40a)
\]

\[
\frac{d \beta}{d \rho} = -\frac{\tan \psi}{\rho}, \quad (40b)
\]

where

\[
\rho = \frac{4\pi e^2 \Gamma}{\rho} r, \quad (41a)
\]

\[
\beta = \frac{B}{4\pi e \Gamma}, \quad (41b)
\]

and the angle \(\psi(\rho)\) determines the direction of the stream of electrons moving outwards.

Eqs. (40) do not have any analytic integrals and cannot be solved in quadratures. We have studied their solutions numerically. They depend on two integration constants, but this time none of them is trivial. Except for special values of these constants, all solutions are confined between two concentric cylinders whose axes lie along the magnetic field. To parametrize the solutions, it is convenient to use the radius of the outer cylinder \(\rho_{\text{max}}\) and the magnitude of the outer magnetic field \(\beta_{\text{out}}\).

Three types of solutions are depicted in figs. 4–6. Again, the velocity angle \(\psi\) and the magnetic field \(\beta\) are shown, but this time as functions of the radial variable \(\rho\). The same radius \(\rho_{\text{max}}\) of the outer cylinder has been chosen in all three cases; they differ in the value of the outer magnetic field \(\beta_{\text{out}}\). There is a close correspondence between the plane and the cylindrical symmetry, which can be noticed by comparing figs. 1–2 and figs. 4–6. In particular, the diagrams 1a (strong field regime) and 2a (weak field regime) resemble the diagrams 4a and 6a, respectively. The differences between the corresponding trajectories may be attributed to the drift which is caused by the field gradient in the radial direction; field \(\beta_{\text{out}}\) outside the larger cylinder is always different from the field \(\beta_{\text{in}}\) inside the smaller cylinder. Diagrams 5a and 5b represent a special situation, in which the inner cylinder disappeared and the whole interior of the larger cylinder is filled. Such a solution exists for every radius of the cylinder. For each \(\beta_{\text{out}}\) there is one value of \(\rho_{\text{max}}\) for which this happens.

In fig. 7 we plot the difference between the radii of the outer and the inner
cylinders, $D = \rho_{\text{max}} - \rho_{\text{min}}$, as a function of the outer field $\beta_{\text{out}}$ for different choices of the radius $\rho_{\text{max}}$. Each curve has two wings; the left wing corresponds to the weak field solution (type 6a), while the right wing corresponds to the strong field solution (type 4a). The two wings always join at a point corresponding to a special solution of the type 3b. For strong fields $\beta_{\text{out}}$, the difference of the two radii becomes independent of $\rho_{\text{out}}$ long before it reaches the zero value.

Even though our equations for $\beta$ and $\psi$ in the case of cylindrical symmetry cannot be explicitly solved, the behavior of $\beta$ and $\psi$ near the boundaries of the confinement region can be described analytically (cf. the appendix). In particular, the behavior of the magnetic field at both boundaries is governed by the square root dependence

$$\beta \approx \beta_0 - C \sqrt{|\rho - \rho_0|},$$

(42)
Fig. 5. Cylindrical symmetry, special case: plasma fills the whole cylinder; $\beta_{\text{out}} = -1.5482342; r_{\text{max}} = 2$. (a) Magnetic field $\beta$ and velocity angle $\phi$ as functions of radial coordinate $\rho$. (b) Individual particle trajectory.

Fig. 6. Cylindrical symmetry, weak field regime ($\beta_{\text{out}} = -1.2; r_{\text{min}} = 2$. (a) Magnetic field $\beta$ and velocity angle $\phi$ as functions of radial coordinate $\rho$. (b) Individual particle trajectory.
where $\rho_o$ is either $\rho_{\max}$ or $\rho_{\min}$. Thus, the derivative of $\beta$ at the boundary is infinite, as in the case of plane symmetry.

Confined, cylindrically symmetric solutions of the Vlasov–Maxwell equations were studied some time ago by Marx. Although his solutions are qualitatively similar to ours, there is an important difference. He assumed the electron distribution function as a product of two (not three as in our case) delta functions. From our point of view, such a distribution function describes a statistical mixture of trajectories which differ in the values of the $r$ and $z$ components of canonical momenta. The two-delta Ansatz leads to the linear (Bessel) equation for the magnetic field, whose solutions differ quantitatively from those described here, especially at the boundaries of the confinement region.
In addition to the confined solutions discussed above, there are also unconfined solutions ($\rho_{\text{max}} \sim \infty$), for which the orbits of individual electrons originate and end at infinity. There is one solution of this type for each $\rho_{\text{min}}$.

5. Concluding remarks

Our treatment of Vlasov–Maxwell equations stresses the particle orbits, which underlie the statistical description; the symmetry of the solution results from the averaging over the orbits which differ only in their positions in space, not in their intrinsic shapes. From this point of view one can look at the configuration with the cylindrical symmetry as a deformation of the configuration with plane symmetry. After all, the hollow cylinder can be obtained by rolling the slab. As is seen in fig. 4b, this bending, accompanied by unbalanced gradients of the magnetic field, results in a drift of electron orbits. One may say that the rolling of the slab results in the compactification of the solution in the $y$ direction. Thus, in order to obtain a fully confined plasma configuration one should perform the second compactification by rolling the cylinder into a torus. However, the corresponding solutions would be much more complicated. There will be a second drift of the orbits associated with the bending of the cylinder and one would have to introduce also an additional current flowing through the center of the torus. This would destroy the cylindrical symmetry of the magnetic field.

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Appendix

Eqs. (17) and (40) for the velocity angle $\psi$ and the magnetic field $\beta$ are singular at the boundaries of the confinement region, because at the turning points $\tan \psi$ is infinite. Since the singularity is the same in both cases, from the knowledge of the analytic solution for the plane symmetry we can find the behavior of the solutions for the cylindrical symmetry near the boundary,
\[ \sin \psi = \pm (1 - A |\rho - \rho_0|), \quad (A.1a) \]
\[ \beta = \beta_0 + C |\rho - \rho_0|^{1/2}, \quad (A.1b) \]

where \( \rho_0 (\beta_0) \) is either \( \rho_{\text{max}} (\beta_{\text{out}}) \) or \( \rho_{\text{min}} (\beta_{\text{in}}) \). The +/- sign in (A.1a) corresponds to the weak/strong field types of solution, respectively.

Upon inserting expressions (A.1) into eqs. (40), we can determine the constants \( A \) and \( C \).

At the outer boundary:

\[ A = -\beta_{\text{out}} - \frac{1}{\rho_{\text{max}}}, \quad (A.2a) \]
\[ C = \frac{1}{\rho_{\text{max}}} \sqrt{\frac{2}{A}}. \quad (A.2b) \]

At the inner boundary:

\[ A = \pm \beta_{\text{in}} + \frac{1}{\rho_{\text{min}}}, \quad (A.3a) \]
\[ C = \pm \frac{1}{\rho_{\text{min}}} \sqrt{\frac{2}{A}}. \quad (A.3b) \]

Let us notice that \( A \) must be positive and hence the magnitude of the outer field must exceed the value \( 1/\rho_{\text{max}} \), needed to produce the cyclotron radius \( \rho_{\text{max}} \).

Formulas (A.1) not only determine the ranges of variation of \( \beta_{\text{out}} \) and \( \beta_{\text{in}} \) with respect to the corresponding radii, but they are also needed to integrate numerically eqs. (40). They are telling us what the first step should look like, when the integration procedure is started at the boundary.

Obviously, eq. (A.3a) and hence our Ansatz (A.1) break down for the special solution, when \( \rho_{\text{min}} = 0 \). This solution fills the whole cylinder and is regular at the origin. One can check by a direct substitution into eqs. (40) that in this case near \( \rho = 0 \) we have

\[ \sin \psi = \beta_0 \frac{\rho}{2}, \quad (A.4a) \]
\[ \beta = \beta_0 \left( 1 + \frac{\rho}{2} \right). \quad (A.4b) \]
Thus, the special solutions form a one-parameter family and are conveniently labelled by the values of the magnetic field $\beta_0$ on the cylinder axis.

References

7) R.C. Davidson, Theory of Nonneutral Plasmas (Benjamin, Reading, 1974), chap. 1.