LEE MODEL AND THE LOCAL FIELD THEORY

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Abstract: The comparison between the Lee model and the usual local field theory is discussed in a certain simple case. The conclusion is that the difficulties associated with the renormalization in the Lee model do not appear in the local field theory.

1. Introduction

The Lee model is the only known model of quantum field theory exactly soluble with renormalization of the coupling constant. However, renormalization of the Lee model in terms of usual concepts of the field theory is impossible if the renormalized coupling constant is greater than a critical coupling constant \(^1\,\,^2\,\,^3\)). In the Lee model without cut-off the critical coupling constant vanishes and thus for any value of the observable coupling constant one must change the fundamental concepts of the field theory. Such a situation in the Lee model implies the possibility that also the usual local field theory is inconsistent for similar reasons \(^4\,\,^5\)). In this paper we want to show that the difficulties which appear in the Lee model are intimately connected with simplifications made in passing from the local field theory to the Lee model. These difficulties disappear when Lee’s interaction Hamiltonian is completed by the lacking terms. We give the comparison of the Lee model with the local field theory for a certain simple case.

2. The Model of the Local Field Theory

Let us consider a simple model of non-relativistic quantum field theory which describes the interaction of the fixed fermions \(V\) and \(N\) with bosons \(\vartheta\). We assume the Hamiltonian for this model in the form

\[
H = H_0 + H_1
= m_0^V \int d^3x V^\dagger(x)V(x) + m_0^N \int d^3x N^\dagger(x)N(x)
+ \frac{1}{2} \int d^3x \left[ \pi^2(x) + (\nabla \varphi(x))^2 + \mu^2 \varphi^2(x) \right] - g_0 \int d^3x \left[ V^\dagger(x)N(x) + N^\dagger(x)V(x) \right] \varphi(x).
\]

With the notations
we can write Hamiltonian (1) in the form

\[ H = H^0 + H^1 = m_o \int d^3 x \psi^\dagger(x) \psi(x) + \frac{1}{2} \int d^3 x [\pi^2(x) + (\nabla \phi(x))^2 + \mu^2 \phi^2(x)] \]

\[ -g_0 \int d^3 x j(x) \phi(x) + \Delta m_o \int d^3 x \psi^\dagger(x) \tau_3 \psi(x). \] (3)

We shall treat the Schrödinger equation with Hamiltonian (3) by means of the perturbation method with respect to the term

\[ H^1 = \Delta m_o \int d^3 x \psi^\dagger(x) \tau_3 \psi(x). \] (4)

It is well known \(^1,6\) that we must find only the physical one-particle states to carry out the renormalization, since the renormalization constants are defined in terms of the matrix elements between these states. We denote the one-particle states in zeroth, first and second approximation by \(|\psi^0(x)\rangle\), \(|\psi^1(x)\rangle\) and \(|\psi^2(x)\rangle\) respectively. We then obtain by usual procedure

\[ |\psi^1(x)\rangle = \left(1 - P \frac{1}{H^0 - m} H^1 \right) |\psi^0(x)\rangle \] (5)

\[ |\psi^2(x)\rangle = N^{-1} \left(1 - P \frac{1}{H^0 - m} H^1 + P \frac{1}{H^0 - m} (H^1 - \Delta E) P \frac{1}{H^0 - m} H^1 \right) |\psi^0(x)\rangle, \] (6)

where

\[ H^0 |\psi^0(x)\rangle = m |\psi^0(x)\rangle, \] (7)

\[ \delta(x - x') N = \langle \psi^0(x) | \left(1 + H^1 P \left(\frac{1}{H^0 - m}\right)^2 \right) H^1 |\psi^0(x')\rangle, \] (8)

\[ \delta(x - x') \Delta E = \langle \psi^0(x) | H^1 |\psi^0(x')\rangle. \] (9)

The vector \(|\psi(x)\rangle\) denotes the column with elements \(|V(x)\rangle\) and \(|N(x)\rangle\) and \(P\) denotes the principal value. From (9) we obtain for energy correction

\[ \Delta E = \tau_3 \Delta m_o \exp(-2g_0^2/g_c^2) = \tau_3 \Delta m. \] (10)

Here \(g_c^2 = [2(2\pi)^3]^{-1} \int d^3 k \omega^{-2} \) and \(\Delta m\) is the observed mass difference in the first approximation. The equation (7) can be solved \(^7\) upon diagonalizing of the Hamiltonian \(H^0\) by means of the unitary transformation

\[ U = \exp \left[ i g_0 \int d^3 x j(x) \frac{1}{\omega^2} \pi(x) \right]. \] (11)

The multiplication by \(\omega^{-2}\) in (11) means multiplication of Fourier transform
of $\pi(x)$. The result of the transformation $U$ is

$$\tilde{H}^0 = UH^0U^{-1} = m\int d^3x\pi^\dagger(x)\pi(x) + \frac{1}{2}\int d^3x[\pi^2(x) + (\nabla\phi(x))^2 + \mu^2\phi^2(x)]$$

$$-\frac{1}{2g_0^2}\int d^3x :j(x):\frac{1}{\omega^2}j(x):.$$  

The colon denotes the Wick normal product and

$$m = m_0 - \delta m = m_0 - \frac{g_0^2}{2(2\pi)^3}\int \frac{d^3k}{\omega^2}.$$  

We can see from (12) that $\pi^\dagger(x)|0\rangle$ is an eigenvector of the operator $\tilde{H}^0$ belonging to the eigenvalue $m$. It follows then that the eigenvector of $H^0$ belonging to the eigenvalue $m$ has the form

$$|\psi^0(x)\rangle = U^{-1}\pi^\dagger(x)|0\rangle.$$  

The renormalized (observed) coupling constant can be obtained from the equation

$$\delta(x-x')\tau_1 = g_0\langle\pi^2(x)|\int d^3x j(x)|\pi^2(x')\rangle.$$  

The renormalization constants $Z_2^V$ and $Z_2^N$ of the fermion fields can be obtained from the equations

$$\delta(x-x')\sqrt{Z_2^V} = \langle 0|V(x)|V^\dagger(x')\rangle,$$

$$\delta(x-x')\sqrt{Z_2^N} = \langle 0|N(x)|N^\dagger(x')\rangle.$$  

Now we find the renormalized coupling constant. From (6), (13) and (15) we have

$$\delta(x-x')\tau_1 = g_0N^{-1}\langle 0|U\left[\int d^3x j(x) + H^1P\frac{1}{H^0-m}\int d^3x j(x)P\frac{1}{H^0-m}H^1\right]U^{-1}|0\rangle.$$  

Using (3), (4) and (11) we find

$$g = g_0N^{-1}[1 - \langle 0|e^{2i\phi_0\tau_1\omega^{-2}\pi(x)}PH_{ob}e^{-2i\phi_0\tau_1\omega^{-2}\pi(x)}|0\rangle],$$  

where

$$H_{ob} = \int d^3k\omega(k)a^\dagger(k)a(k).$$  

With the abbreviations

$$a(g_0^2) = \langle 0|e^{2i\phi_0\tau_1\omega^{-2}\pi(x)}PH_{ob}e^{-2i\phi_0\tau_1\omega^{-2}\pi(x)}|0\rangle = \sum_{n=1}^\infty \frac{1}{n!}\left[\frac{g_0^2}{2(2\pi)^3}\right]^n\int\frac{d^3k_1}{\omega_1^3}\cdots\frac{d^3k_n}{\omega_n^3}\frac{1}{(\omega_1 + \cdots + \omega_n)^2},$$
we obtain finally

\[ g = \frac{1 - (\Delta m)^2 a(g_0^2)}{1 + (\Delta m)^2 a(g_0^2)} \approx g_0 \left( 1 - 2(\Delta m)^2 a(g_0^2) \right). \] (21)

It can be shown that the power series (20) is convergent for \( g_0^2 < g_{\text{max}} \) where \( g_{\text{max}} > (2\pi)^2 e^{1/2} \). From (15) and (17) we can obtain the renormalization constants \( Z^V_2 \) and \( Z^N_2 \):

\[ \sqrt{Z^V_2} = \exp\left(-g_0^2/2g_{\text{cr}}^2\right)N^{1/2}\left[1 - \Delta mb_1(g_0^2) + (\Delta m)^2 b_2(g_0^2)\right], \] (22)

\[ \sqrt{Z^N_2} = \exp\left(-g_0^2/2g_{\text{cr}}^2\right)N^{1/2}\left[1 + \Delta mb_1(g_0^2) + (\Delta m)^2 b_2(g_0^2)\right]. \] (23)

The functions \( b_1(g_0^2) \) and \( b_2(g_0^2) \) are also defined by series which are convergent like (20) in a certain neighbourhood of \( g_0^2 = 0 \).

The vertex part renormalization constant \( Z_1 \) can be find from the well-known Dyson equation

\[ g = Z_1^{-1}(Z^V_2 Z^N_2)^{1/2} Z_3^{1/2} g_0. \] (24)

The boson field renormalization constant \( Z_3 \) is equal to 1 and we have from (24)

\[ Z_1 = \exp\left(-g_0^2/2g_{\text{cr}}^2\right)[1 + (\Delta m)^2 C(g_0^2)], \] (25)

where

\[ C(g_0^2) = b_1^2(g_0^2) + 2b_2(g_0^2) + a(g_0^2). \] (26)

### 3. The Lee Model

The Hamiltonian for the Lee model 1) has the form

\[ H^L = H_0 + H^L_1 = m_0^V \int \! d^3 x V^\dagger(x)V(x) + m_0^N \int \! d^3 x N^\dagger(x)N(x) \]

\[ + \frac{1}{2} \int \! d^3 x \left[ \pi^2(x) + (\nabla \phi(x))^2 + \mu^2 \phi^2(x) \right] \]

\[ - g_0 \int \! d^3 x \left[ V^\dagger(x)N(x)\phi^{(+)}(x) + N^\dagger(x)V(x)\phi^{(-)}(x) \right]. \] (27)

The renormalization constants in this model are

\[ Z_1 = 1, \quad Z_3 = 1, \quad Z_2^N = 1, \quad (Z^V_2)^{-1} = 1 + \frac{g_0^2}{2(2\pi)^3} \int \! \frac{d^3 k}{\omega} \frac{1}{(\omega - \Delta m)^2}. \] (28)

The renormalized coupling constant defined by (25) has the form

\[ g = \left[ 1 + \frac{g_0^2}{2(2\pi)^3} \int \! \frac{d^3 k}{\omega} \frac{1}{(\omega - \Delta m)^2} \right]^{1/2} g_0. \] (29)

It is well known that the equation (29) is the source of all difficulties in the Lee model. In fact, no real solution of this equation exists for \( g_0 \) if \( g \) is finite.
4. Discussion

The comparison between the Lee model and the model described in § 2 can be carried out in terms of Feynman diagrams. In the model of local field theory described by the Hamiltonian (1) there is an infinite number of self energy and vertex part diagrams. The first few diagrams are the following

\[ (30) \]

\[ (31) \]

The symmetry between self energy and vertex part diagrams leads to the reduction of the infinite parts of renormalization constants \( Z_1 \) and \( Z_2 \) in the product \( Z_1^{-1}(Z_2^V Z_2^N)^4 \). In the particular case of equal masses \(^7,^8\) we have simply the Ward identity \( Z_1^{-1} Z_2^{-1} = 1 \).

In the Lee model this symmetry is disturbed. One diagram appears here as the correction to the self energy but there are no radiation corrections to the vertex part. In the Lee model instead of (30) and (31) we have

\[ (32) \]

The infinite parts of the constants \( Z_1 \) and \( Z_2 \) cannot reduce and this leads to the known difficulties in the renormalization procedure.

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