

Gaussons: Solitons of the Logarithmic Schrödinger Equation

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Abstract

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Properties of the Schrödinger equation with the logarithmic nonlinearity are briefly described. This equation possesses soliton-like solutions in any number of dimensions, called gaussons for their Gaussian shape. Excited, stationary states of gaussons of various symmetries, in two and three dimensions are found numerically. The motion of gaussons in uniform electric and magnetic fields is studied and explicit solutions describing linear and rotational internal oscillations are found and analyzed.

1. Introduction

It has been shown by us several years ago [1, 2] that the Schrödinger equation with the logarithmic nonlinearity (LSE) in any number of dimensions possesses many distinct features which make this equation unique among nonlinear wave equations. A description of nonlinear wave mechanics based on the LSE has been presented in [2], where we also have proved various mathematical properties of this equation. One of the most remarkable features of the LSE is the existence of analytic, soliton-like solutions in any number of dimensions, called by us gaussons for their Gaussian shape. Numerical studies of gaussons have shown [3] that they are stable under collisions in one and two dimensions over a wide range of energies. However, in a narrow energy region a third gausson may be produced, depending on the phase relation between the two colliding gaussons. Several studies of both nonrelativistic and relativistic wave equations with the logarithmic nonlinearity have been also published recently by other authors [4-8].

In the present paper we shall continue our investigation of gaussons, but for completeness we shall also include a brief summary of main results obtained earlier (Section 2). In Section 3 we study excited stationary states of a free gausson. Section 4 is devoted to an analysis of the center of mass motion and the internal motions of gaussons in an external electromagnetic field. Exact analytic solutions are given for an arbitrary configuration of electric and magnetic fields.

2. Basic properties of the LSE

The logarithmic Schrödinger equation in the configuration space of N particles, in the absence of external forces has the form:

$$i\hbar\partial_t\psi(r_1, \dots, r_N, t) = \left[-\hbar^2 \sum_{k=1}^N \frac{1}{2m_k} \Delta_k - b \ln(|\psi|^2 a^{3N}) \right] \psi(r_1, \dots, r_N, t) \quad (1)$$

In dimensionless notation, eq. (1) reads:

$$i\partial_t\phi(r, t) = [-\Delta - \ln|\phi|^2]\phi(r, t) \quad (2)$$

where

$$r = \hbar^{-1}\sqrt{2b}(r_1\sqrt{m_1}, \dots, r_N\sqrt{m_N}) \quad (3)$$

$$\phi = a^{3N/2}\psi, \quad (4)$$

and time is measured in units \hbar/b . The Laplacian in eq. (2) stands for the sum of second derivatives in $n = 3N$ dimensions. In what follows n will stand for an arbitrary natural number, not necessarily a multiple of 3.

Even though eq. (2) is nonlinear, it has the property of allowing for the separation of variables in a manner which is characteristic of linear theories. Namely, we may construct solutions in n dimensions by taking any product of solutions in n_1 and n_2 ($n_1 + n_2 = n$) dimensions.

Without any loss of generality we may assume that solutions of eq. (2) are normalized to 1,

$$\int d^n x |\phi(r, t)|^2 = 1 \quad (5)$$

exactly as in a linear theory. Any other solution can be reduced to a normalized solution by the following simultaneous change of its norm and its phase:

$$\phi(r, t) \rightarrow \|\phi\|^{-1} \exp(-it \ln \|\phi\|^2) \phi(r, t) \quad (6)$$

Eq. (2) has $1 + 1 + n + n(n-1)/2$ constants of the motion:

the norm

$$\|\phi\|^2 = \int d^n x \phi^* \phi \quad (7)$$

the energy

$$E[\phi] = \int d^n x (\nabla\phi^* \nabla\phi - \phi^* \phi \ln|\phi|^2) \quad (8)$$

the linear momentum

$$P_i[\phi] = -i \int d^n x \phi^* \nabla_i \phi, \quad (9)$$

and the angular momentum

$$M_{ij}[\phi] = -i \int d^n x \phi^* (x_i \nabla_j - x_j \nabla_i) \phi \quad (10)$$

For all functions ϕ of a given norm, the energy integral is bounded from below. For functions normalized to unity, which we consider here, one obtains the following lower bound for the energy:

$$E[\phi] \geq E_0 = n(1 + \frac{1}{2} \ln \pi) \tag{11}$$

This bound is attained only on the Gaussian functions (apart from a constant phase factor) of the form:

$$\phi(\mathbf{r}) = \pi^{-n/4} \exp [-(\mathbf{r} - \boldsymbol{\eta})^2/2] \tag{12}$$

Owing to the invariance under Galilean transformations, we can use the static solutions (12) of the LSE to generate uniformly moving solutions characterized by momentum \mathbf{p} and initial position $\boldsymbol{\eta}$:

$$\phi(\mathbf{r}, t) = \pi^{-n/4} \exp [-i(E_0 + \mathbf{p}^2)t + i\mathbf{p} \cdot \mathbf{r}] \exp [-(\mathbf{r} - \boldsymbol{\eta} - 2\mathbf{p}t)^2/2] \tag{13}$$

We call these solutions the gaussons.

Frequencies ω of stationary solutions of eq. (2),

$$\phi(\mathbf{r}, t) = e^{-i\omega t} \phi_\omega(\mathbf{r}) \tag{14}$$

obey the Planck relation

$$\omega = E[\phi_\omega] \tag{15}$$

again a special feature of the logarithmic nonlinearity. Stationary solutions obey also the following simple version of the virial theorem:

$$\int d^n x \nabla_i \phi^* \nabla_j \phi = \frac{1}{2} \delta_{ij} \tag{16}$$

The spectrum of small oscillations of a gausson is real and purely discrete. Therefore the gaussons are stable, since their small oscillations remain small for all times.

3. Excited states of a free gausson

The ground state of a free gausson in any number of dimensions is described by the Gaussian wave function (12). A natural question which arises here is, whether there exist excited stationary states of gaussons. It has been shown in [2] that there are no such states in one dimension. In more than one dimension we have no analytic solutions, but we report here the existence of several families of excited states of different symmetries, which we found numerically. We shall restrict ourselves in this paper to two and three dimensions.

The simplest case is clearly that of full rotational symmetry of the wave function. The LSE in this case has the form:

$$\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + \ln |\phi|^2 + E \right) \phi = 0 \tag{17}$$

where n is equal to 2 or 3. We may eliminate the parameter E from this equation by the following rescaling of ϕ :

$$\phi(r) = e^{-E/2} f(r) \tag{18}$$

which results in the change of the normalization:

$$\|f\|^2 = e^E \quad E = 2 \ln \|f\| \tag{19}$$

The equation for f has a fairly simple form:

$$\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + 2 \ln |f| \right) f = 0 \tag{20}$$

All global solutions of this equation are real, up to a constant

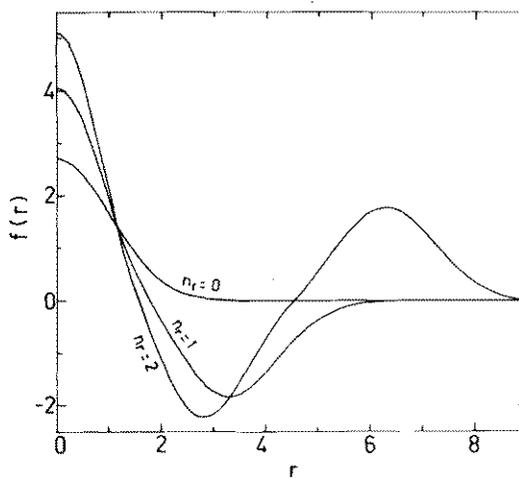


Fig. 1. Wave functions of rotationally symmetric stationary states in 2 dimensions.

phase factor. This follows from the continuity equation, which has here the same form as in the linear theory [2].

We solved eq. (20) numerically by the shooting method with the use of the HP programmable pocket calculator.

Because of the presence of the factor $(n-1)/r$, which is singular, the derivative of f must vanish at the origin, if f is to be regular. We carried out the search for solutions which go to zero at infinity by varying the value of f at the origin. In Figures 1 and 2 we show the plots of f for the three lowest lying states in two and three dimensions. These states are labelled by the quantum number n_r — the number of zeros of the wave function. The states corresponding to $n_r = 0$ are described by the Gaussian wave functions

$$f = \exp(n/2 - r^2/2) \tag{21}$$

therefore (cf. eq. (11)), they are the ground states. In Table I we give the values f_0 of f at the origin, the norm of f , and the energies relative to the ground state calculated from the formulas (11) and (19) for the three lowest lying states. We believe that for each n there are infinitely many rotationally symmetric excited states.

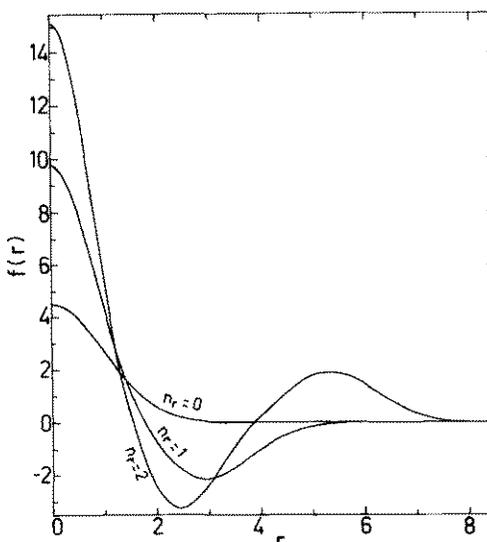


Fig. 2. Wave functions of rotationally symmetric stationary states in 3 dimensions.

Table I. The values of the wave function at the origin f_0 , the norms of the wave functions $\|f\|$, and the relative energies with respect to the ground state calculated numerically for the three lowest, rotationally symmetric states in 2 and 3 dimensions

n	2			3			
	n_r	0	1	2	0	1	2
f_0		2.72	4.04	5.09	4.48	9.72	15.09
$\ f\ $		4.82	12.00	19.07	10.58	31.31	58.07
$E - E_0$		0.00	1.83	2.75	0.00	2.17	3.41

In addition to rotationally symmetric solutions, we also found in two dimensions solutions with non zero angular momentum, corresponding to the values of the magnetic quantum number $m = \pm 1$. Wave functions describing these solutions have the form:

$$\phi(r, \varphi) = e^{-E/2} e^{im\varphi} f(r) \tag{22}$$

where f is real and obeys the equation:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + 2 \ln |f| \right) f = 0 \tag{23}$$

For all regular solutions of this equation, f must behave near the origin as

$$f(r) \simeq f'_0 r \tag{24}$$

In this case we varied the values of f'_0 , and the results of our numerical calculations are shown in Fig. 3 and in Table II.

It is worth noting that the energy required to split a gausson in its ground state into two gaussons of norm $1/\sqrt{2}$ each is merely $\ln 2 \simeq 0.69$. Therefore, the energies of excited states are relatively large.

In order to understand the meaning of our numerical results, we shall use the standard mechanical analogy. To this end, we shall treat eqs. (20) or (23) as the equation of motion for a fictitious point particle whose position is given by f , and with r playing a role of the time variable. The nonlinear part of the equation represents a force derived from the potential $V(f)$,

$$V(f) = \frac{1}{2} f^2 (\ln f^2 - 1) \tag{25}$$

shown in Fig. 4. The first derivative term corresponds to friction, with the friction coefficient decreasing with time. Finally, the term m^2/r^2 in eq. (23) represents an additional repulsive force rapidly decreasing with time. In this mechanical picture, the problem of finding stationary states is that of finding such initial data (position and velocity), that the particle ends its motion after infinite time at the top of the potential barrier at

Table II. The values of the derivative at the origin f'_0 , the norms of the wave functions $\|f\|$, and the relative energies with respect to the ground state calculated numerically for the three lowest $|m| = 1$ states in two dimensions

n_r	1	2	3
f'_0	1.51	2.50	3.34
$\ f\ $	8.92	16.01	23.02
$E - E_0$	1.23	2.40	3.13

$f = 0$. Owing to friction, all remaining motions will end up at the bottom of one of the two valleys at $f = \pm 1$. Last stages of those motions consist of small, damped oscillations around ± 1 with the period $\pi\sqrt{2}$ whose amplitudes decrease as $r^{-(n-1)/2}$. In Fig. 5 we show how the solutions of eq. (20) for $n = 3$ change when the initial value of the wave function is changed.

Our numerical results were obtained for $n = 2, 3$, but we expect quantitatively the same behavior in any number of dimensions. Of course, with the help of the separation of variables method, we may add new dimensions in a trivial manner by multiplying a solution in 2 or 3 dimensions by a Gaussian (or any other) solution in the remaining dimensions.

4. Gaussons in a uniform electromagnetic field in 3 dimensions

In a uniform electromagnetic field, for any nonlinear Schrödinger equation of the form:

$$i\hbar \partial_t \psi(r, t) = \left[\frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{e}{2c} \mathbf{B} \times \mathbf{r} \right)^2 - e\mathbf{E} \cdot \mathbf{r} + F(|\psi|^2) \right] \psi(r, t) \tag{26}$$

the center of mass motion is the same as for the classical particle (Ehrenfest theorem, cf. [2]), and can be completely separated from the internal motion. In order to show this explicitly, we will write the solution of eq. (26) in the form:

$$\psi(r, t) = \exp \left[-\frac{i}{\hbar} \int_0^t dt \left(\frac{m}{2} \dot{\xi}^2 + \frac{e}{2c} (\xi \times \dot{\xi}) \cdot \mathbf{B} \right) + \frac{i}{\hbar} \mathbf{p} \cdot \mathbf{r} \right] \phi(\mathbf{p}, t) \tag{27}$$

where $\mathbf{p} = \mathbf{r} - \xi(t)$ and $\xi, \dot{\xi}$ and \mathbf{p} are the position, velocity and canonical momentum, i.e., they fulfill the equations of classical mechanics:

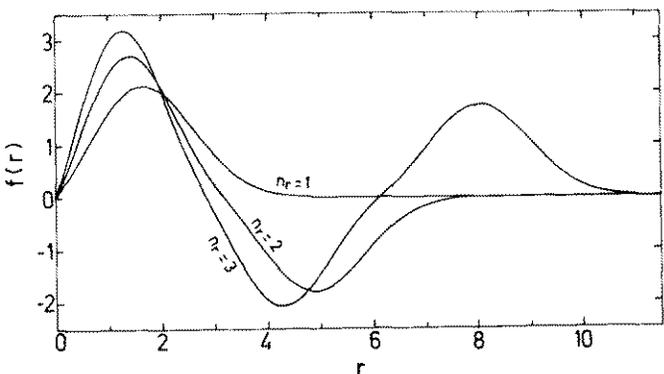


Fig. 3. Wave functions of stationary states with $m = \pm 1$ in 2 dimensions.

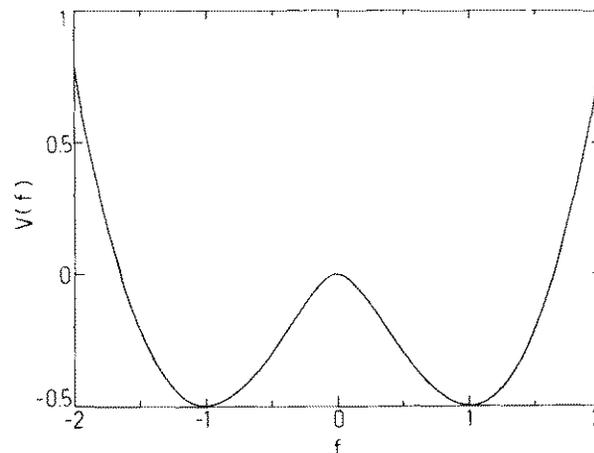


Fig. 4. Potential for the mechanical analog of the LSE.

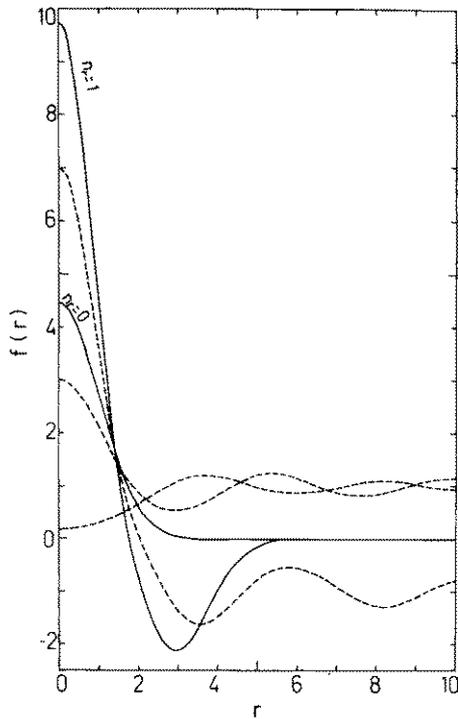


Fig. 5. Illustration of the shooting method used to obtain rotationally symmetric stationary solutions of the LSE in 3 dimensions.

$$m\dot{\xi} = e(E + \dot{\xi}/c \times B) \tag{28}$$

$$p = m\dot{\xi} - \frac{e}{2c} \xi \times B \tag{29}$$

Simple calculations lead to the following equation for ϕ :

$$i\hbar\partial_t\phi(\mathbf{p}, t) = \left[\frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{e}{2c} \mathbf{B} \times \mathbf{p} \right)^2 + F(|\phi|^2) \right] \phi(\mathbf{p}, t), \tag{30}$$

where the space derivatives act on \mathbf{p} , but the time derivative does not. Thus, the electric field has been completely eliminated from the wave equation. If eq. (26) possesses localized solutions, then they move as a whole as classical particles under the influence of the Lorentz force. In addition to the center of mass motion, such localized solutions, in general, will undergo complicated internal motions described by eq. (30). In this section we shall study a family of internal motions for the logarithmic nonlinearity. Equation (30) may be written in this case in the following dimensionless form:

$$i\partial_\tau\phi(\xi, \tau) = \left[\frac{1}{2} (-i\nabla + \kappa\hat{\Omega} \cdot \xi)^2 - \ln|\phi| \right] \phi(\xi, \tau) \tag{31}$$

where

$$\tau = 2bt/\hbar \tag{32}$$

$$\xi = \hbar^{-1} \sqrt{2mb} \mathbf{p} \tag{33}$$

$$\kappa = e\hbar|B|/4mbc \tag{34}$$

and the antisymmetric matrix $\hat{\Omega}$ is defined in terms of the unit vector \mathbf{n} pointing in the direction of \mathbf{B} :

$$\Omega_{ij} = \epsilon_{ijk} n_k \tag{35}$$

We shall seek solutions of eq. (31) in the generalized Gaussian form, so that they will describe internal motions of a gaussian:

$$\phi(\xi, \tau) = (\det A)^{1/4} \pi^{-3/4} \exp \left[i\varphi(\tau) - \frac{1}{2} \sum_{i,j} (A(\tau) + iB(\tau))_{ij} \xi_i \xi_j \right] \tag{36}$$

Upon substituting (36) into eq. (31), we obtain the following set of ordinary differential equations for the matrices A and B and the phase φ :

$$\dot{A} = AB + BA + \kappa[\Omega, A] \tag{37}$$

$$\dot{B} = B^2 - A^2 + A + \kappa[\Omega, B] - \kappa^2 \Omega^2 \tag{38}$$

$$\dot{\varphi} = -\frac{3}{4} \ln \pi - \frac{1}{2} \text{tr} A + \frac{1}{4} \ln (\det A) \tag{39}$$

Equivalent equations were derived in [2] in the absence of the magnetic field.* Equation (39) for the phase does not contain any interesting information about the motion, and it may be always solved by a straightforward integration, once the matrix A is found.

We shall describe here solutions in which the dependence of ϕ on z , measured in the direction of \mathbf{B} , separates from the dependence on x and y . This means that the elements with indices 13, 23, 31, and 32 in matrices A and B are zero. The equations for elements with indices 33 in the matrices A and B do not involve the magnetic field and were solved before (cf. eqs. (6.23–28) of [2]). Their solutions describe periodic changes of the Gaussian radius in the z direction.

Thus we have reduced the problem to that of finding 2×2 matrices A and B satisfying eqs. (37) and (38), where now

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{40}$$

We found two types of explicit solutions of those equations, corresponding to different types of internal motions of a gaussian.

Solutions of the first type describe oscillations of the gaussian resulting in periodic changes of its radius in the direction perpendicular to \mathbf{B} . Solutions of the second type describe rotational motions of elliptical deformations of the gaussian in the xy -plane.

In order to obtain solutions of the oscillatory type, we choose the matrices A and B in the form:

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \tag{41}$$

$$B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \tag{42}$$

From eqs. (37) and (38) we obtain the following differential equations for the eigenvalues a and b :

$$\dot{a} = 2ab \tag{43}$$

$$\dot{b} = b^2 - a^2 + a + \kappa^2 \tag{44}$$

In the same manner as in [2] we shall use the energy integral for eqs. (43) and (44):

$$\epsilon = a + (b^2 + \kappa^2)a^{-1} - \ln a \tag{45}$$

in order to solve these equations. Upon solving (45) for b and substituting the result into (43), we obtain a single equation for

* Our new parameter τ is twice that of [2]. In eq. (6.16) of [2], $-n/2 \ln \pi$ was erroneously written as n .

$a,$
 $\dot{a} = 2a(a \ln a + \epsilon a - a^2 - \kappa^2)^{1/2}$ (46)

It is clear that solutions of this differential equation exist only when ϵ is sufficiently large relative to κ , so that the expression under the square root is positive for some value of a . As a function of time, a will change periodically between the values a_{\min} and a_{\max} , similarly as for the oscillations in the z -direction. When $a_{\min} = a_{\max}$, no oscillations may occur, and we obtain the static solution found already in [2]:

$a = \text{const.} = \frac{1}{2}(\sqrt{1 + 4\kappa^2} + 1)$ (47)

This is the lowest energy state among all states of the oscillatory type and its energy is:

$E_{\perp} = \ln \pi + 2\epsilon = \ln \pi + \int \sqrt{1 + 4\kappa^2} - \int \ln \frac{\sqrt{1 + 4\kappa^2} + 1}{2} + 1$ (48)

We have denoted it by E_{\perp} , since it represents only the energy in the plane perpendicular to B .

Solutions of the second type are most easily obtained after the transformation of eqs. (37) and (38) to a new coordinate system, rotating with frequency ω (measured in units of τ^{-1}) around the field direction. We choose the orientation of this rotation to be the same as in the cyclotron motion. The generator of such rotations is the matrix Ω , so that the derivatives in the laboratory system are related to those in the rotating frame by the formula:

$\dot{A} = (\dot{A})_{\text{rot}} + \omega[\Omega, A]$ (49)

In the rotating frame, time independent solutions are to be determined from the equations:

$0 = AB + BA - (\omega - \kappa)[\Omega, A]$ (50)

$0 = B^2 - A^2 + A - (\omega - \kappa)[\Omega, B] - \kappa^2 \Omega^2$ (51)

Since both matrices A and B are symmetric, they can be parameterized in terms of two scalar parameters α and β and two vector parameters α and β :

$A = \begin{pmatrix} \alpha + \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha - \alpha_1 \end{pmatrix}$ (52)

$B = \begin{pmatrix} \beta + \beta_1 & \beta_2 \\ \beta_2 & \beta - \beta_1 \end{pmatrix}$ (53)

The only restriction on the values of those parameters comes from the positive definiteness of A :

$\alpha > |\alpha|$ (54)

Equations (50) and (51) now read:

$\alpha\beta + \beta\alpha + (\omega - \kappa)n \times \alpha = 0$ (55)

$(\frac{1}{2} - \alpha)\alpha + \beta\beta + (\omega - \kappa)n \times \beta = 0$ (56)

$\beta^2 + \beta^2 - \alpha^2 - \alpha^2 + \alpha + \kappa^2 = 0$ (57)

$\alpha\beta + \alpha \cdot \beta = 0$ (58)

The solution of these equations becomes simple, if we first observe that $\beta = 0$ and $\alpha \cdot \beta = 0$. This follows from eqs. (55) and (58). Eq. (55) multiplied by α gives:

$\alpha\alpha \cdot \beta + \beta\alpha^2 = 0$ (59)

Upon subtracting (59) from (58) multiplied by α , we obtain:

$\beta(\alpha^2 - \alpha^2) = 0$ (60)

The case $\alpha^2 = \alpha^2$ is excluded by the condition (54). Thus, we must have $\beta = 0$ and from (58) we also have $\alpha \cdot \beta = 0$.

The simplified set of equations reads:

$\alpha\beta + (\omega - \kappa)n \times \alpha = 0$ (61)

$(\frac{1}{2} - \alpha)\alpha + (\omega - \kappa)n \times \beta = 0$ (62)

$\beta^2 - \alpha^2 - \alpha^2 + \alpha + \kappa^2 = 0$ (63)

Taking the vector product of eq. (61) with n , we obtain

$-(\omega - \kappa)\alpha + \alpha n \times \beta = 0$ (64)

The compatibility condition for eqs. (64) and (62) is:

$(\frac{1}{2} - \alpha)\alpha + (\omega - \kappa)^2 = 0$ (65)

which we can solve for α ,

$\alpha = \frac{1}{4}(\sqrt{1 + 16(\omega - \kappa)^2} + 1)$ (66)

The negative root is excluded by (54). From eq. (61) we may now determine β ,

$\beta = -\alpha^{-1}(\omega - \kappa)n \times \alpha$ (67)

The length of α is found from (67) and (63),

$\alpha^2 = \alpha^2 - 2\alpha((\omega - \kappa)^2 - \kappa^2)$ (68)

The direction of α is arbitrary, since it determines the initial orientation of the deformation ellipse.

From (68) and (54) follows the following restriction on ω :

$(\omega - \kappa)^2 > \kappa^2$ (69)

The second restriction on ω ,

$\alpha \geq 2((\omega - \kappa)^2 - \kappa^2)$, (70)

follows from an obvious condition $\alpha^2 \geq 0$. After some algebraic transformations, these two restrictions may be written in the form:

$(\kappa^2 + \frac{1}{4} + \frac{1}{2}\sqrt{\kappa^2 + \frac{1}{4}})^{1/2} \geq |\omega - \kappa| > \kappa$ (71)

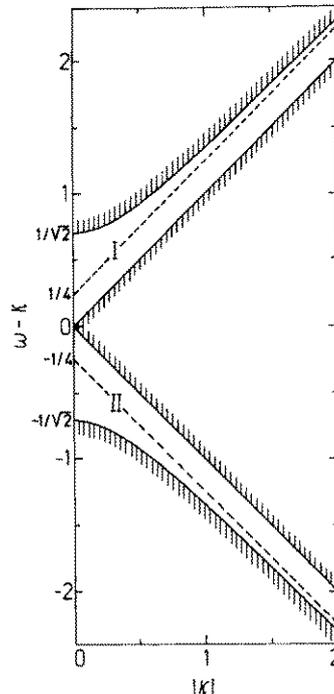


Fig. 6. Two allowed regions for the frequency ω of the rotating deformation of a gausson in a uniform magnetic field.

The allowed regions for ω are shown in Fig. 6. The energy of rotationally deformed gaussons is:

$$E_{\perp} = \ln \pi + \frac{(\omega - \kappa)(2\omega - 3\kappa)}{(\omega - \kappa)^2 - \kappa^2} + \frac{-8\kappa(\omega - \kappa)}{1 + \sqrt{1 + 16(\omega - \kappa)^2}} - \frac{1}{2} \ln \frac{1 + \sqrt{1 + 16(\omega - \kappa)^2}}{2} - \frac{1}{2} \ln ((\omega - \kappa)^2 - \kappa^2) \quad (72)$$

To analyze our results, we shall assume that $\kappa \geq 0$. Then the upper region I of allowed values of ω contains frequencies which are larger than the cyclotron frequency 2κ . For strong magnetic fields, the ratio $\omega/2\kappa$ tends to 1. The lower region II contains frequencies ω of the order of 1 (in units $2b/\hbar$). For strong magnetic fields, the restrictions on ω become independent of κ : $0 > \omega \geq -1/4$. At zero magnetic field, ω ranges from $-1/\sqrt{2}$ to $1/\sqrt{2}$. The states lying on the curved boundaries of regions I and II are exactly those described by the formulas (47) and (48). Near these boundaries there lie states which are nearly circular, with small elliptical, rotating deformations. At zero magnetic field, these solutions go into small quadrupole oscillations, contained in a general family of small oscillations of a free gausson,* described in [2] (cf. eqs. (6.31) and (6.32) of [2]). Near the boundaries $|\omega - \kappa| = |\kappa|$ there lie states of a gausson whose ellipse is stretched out almost into a

* Notice that the frequency of periodic changes of the coefficients of a quadratic form, rotating with the angular frequency ω , is equal to 2ω .

one dimensional segment, rotating with the cyclotron frequency in the region I and almost stationary in the region II. In the absence of magnetic field, in addition to rotational oscillations, we may also have linear oscillations in any direction described in [2].

5. Outlook

New analytic solutions of the logarithmic Schrödinger equation, described in this paper, add to an already imposing list of exact results which can be derived for this nonlinear equation. It is remarkable that most results hold in any number of dimensions. Even if all physical reasons which motivated our studies in [2] are invalid, the LSE will still offer a very useful testing ground for testing various new ideas about nonlinear wave equations.

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