Gauge-invariant Variables in the Yang-Mills Theory

by

I. BIAŁYNICKI-BIRULA

Presented by L. INFELD on January 18, 1963

1. In two recent papers, Mandelstam [1] proposed a new scheme for quantizing electrodynamics and general relativity. In his gauge-independent formulation of quantum electrodynamics the quantities

\begin{align*}
F_{\mu\nu} &= A_{\mu,\nu} - A_{\nu,\mu}, \\
\Phi(x, P) &= \exp\left(ie \int_{-\infty}^{\infty} d\xi^{\mu} A_{\mu}(\xi)\right) \varphi, \\
\Phi^*(x, P) &= \exp\left(-ie \int_{-\infty}^{\infty} d\xi^{\mu} A_{\mu}(\xi)\right) \varphi
\end{align*}

are taken as basic field variables. The linear integrals in Eqs. (2) and (3)

\begin{align*}
\int_{-\infty}^{\infty} d\xi^{\mu} A_{\mu}(\xi) &= \int_{-\infty}^{\infty} d\tau \frac{d\xi^{\mu}}{d\tau} A_{\mu}(\tau)
\end{align*}

are evaluated along certain path \( P : \xi^\nu(\tau) \). The fields \( \Phi \) and \( \Phi^* \) will depend in general on this path which has been explictely indicated in formulas (2) and (3). The field equations expressed in terms of gauge-invariant variables have the form

\begin{align*}
(\Box + m^2) \Phi &= 0, \\
(\Box + m^2) \Phi^* &= 0, \\
F_{\nu,\nu} + j^\mu &= 0, \\
\dot{j}^\mu &= ie (\Phi^* \delta^\mu \Phi - \delta^\mu \Phi^* \cdot \Phi).
\end{align*}

Potentials do not appear in these equations.

[135]
2. In this paper we apply Mandelstam’s ideas to the Yang—Mills field [2]. For simplicity, we consider only a pure Yang—Mills field. The inclusion of other fields presents no additional difficulties, since the Yang—Mills field alone interacts with itself. The Lagrangian and the field equations for the Yang—Mills potentials $\tilde{b}_\mu$ have the form

$$L = -\frac{1}{4} \tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu},$$

(9)

$$\tilde{f}_{\mu\nu} - 2i e \tilde{b}_\nu \times \tilde{f}^{\mu\nu} = 0,$$

(10)

$$\tilde{f}_{\mu\nu} = \tilde{b}_{\mu,\nu} - \tilde{b}_{\nu,\mu} + 2ie \tilde{b}_\mu \times \tilde{b}_\nu.$$

(11)

It is convenient to use matrix variables [2]

$$F_{\mu\nu} = \tilde{f}_{\mu\nu} \cdot \tilde{\tau}, \quad B_\mu = \tilde{b}_\mu \cdot \tilde{\tau},$$

where $\tilde{\tau}$ are the three Pauli matrices. In terms of these variables Eqs. (10) and (11) read

$$F_{\mu\nu} + i e [B_\nu, F^{\mu\nu}] = 0,$$

(13)

$$F_{\mu\nu} = B_{\mu,\nu} \mp B_{\nu,\mu} - i e [B_\mu, B_\nu].$$

(14)

Under the gauge transformation

$$S = \exp (-i e \lambda (x)),$$

(15)

where $\lambda (x) = \tilde{\lambda} (x) \cdot \tilde{\tau}$, the field variables $B_\mu$ and $F_{\mu\nu}$ transform as follows

$$'B_\mu = SB_\mu S^{-1} + ie^{-1} \partial_\mu S \cdot S^{-1},$$

(16)

$$'F_{\mu\nu} = SF_{\mu\nu} S^{-1}.$$

(17)

In contrast to the electromagnetic case the field strength $F_{\mu\nu}$ is not invariant under the gauge transformations.

3. We will construct in this paragraph the gauge-invariant field $(\mathcal{F}_{\mu\nu})$. Let $U(x,P)$ be the following path-dependent matrix

$$U(x,P) = T \exp \left(-ie \int_{-\infty}^{\tau} B_\mu d\xi^\mu \right) = T \exp \left(-ie \int_{-\infty}^{\tau} B_\mu (\tau) \frac{d\xi^\mu}{d\tau} d\tau \right),$$

(18)

where $T$ denotes the $\tau$-ordering of $B_\mu$’s. The matrices $B_\mu (\tau)$ with larger values of $\tau$ stand to the left with respect to those with smaller $\tau$. Following Mandelstam we define derivatives of the $U$ matrix by the formula

$$\partial_\mu U(x,P) = \lim_{dx \to 0} \frac{U(x + dx, P') - U(x, P)}{dx},$$

(19)
where the paths $P'$ and $P$ differ by $dx^\mu$. With the use of (18) we obtain

\begin{equation}
\partial_\mu U(x, P) = -ie \, B_\mu(x) \, U(x, P).
\end{equation}

The hermitian conjugate matrix $U^+$ can be written in the form

\begin{equation}
U^+(x, P) = \bar{T} \exp \left( ie \int_{-\infty}^{x} B_\mu \, d\xi^\mu \right),
\end{equation}

where $\bar{T}$ denotes the anti-$\tau$-ordering of $B_\mu$'s. $U^+$ obeys the equation

\begin{equation}
\partial_\mu U^+(x, P) = ieU^+(x, P) \, B_\mu(x).
\end{equation}

The gauge-invariant field $\mathcal{F}_{\mu\nu}$ is a path-dependent quantity and has the form

\begin{equation}
\mathcal{F}_{\mu\nu}(x, P) = U^+(x, P) \, F_{\mu\nu} \, U(x, P).
\end{equation}

As a result of Eqs. (10) and (11) it obeys the following equations

\begin{equation}
\mathcal{F}_{\mu\nu} = 0,
\end{equation}

\begin{equation}
\mathcal{F}_{\mu\nu, \lambda} + \mathcal{F}_{\lambda\mu, \nu} + \mathcal{F}_{\nu\lambda, \mu} = 0.
\end{equation}

We may also introduce path-dependent potentials. The field equations (25) guarantee the existence of the vector field $B_\mu$ which is linked to $\mathcal{F}_{\mu\nu}$ by the formula

\begin{equation}
\mathcal{F}_{\mu\nu} = B_{\mu, \nu} - B_{\nu, \mu}.
\end{equation}

4. The invariance of $\mathcal{F}_{\mu\nu}$ under the gauge transformations can be proved with the use of Eqs. (20) and (22) for the $U$ matrix. The gauge-transformed $\mathcal{F}_{\mu\nu}$ is given by the formula

\begin{equation}
'\mathcal{F}_{\mu\nu} = 'U^+ \, 'F_{\mu\nu} \, 'U = 'U^+ \, SF_{\mu\nu} \, A^{-1} \, 'U,
\end{equation}

where

\begin{equation}
'U = T \exp \left( -ie \int_{-\infty}^{x} 'B_\mu \, d\xi^\mu \right),
\end{equation}

\begin{equation}
= T \exp \left[ -ie \int_{-\infty}^{x} (SB_\mu \, S^{-1} + ie^{-1} \partial_\mu S \cdot S^{-1}) \, d\xi \right].
\end{equation}

The gauge-transformed $U$ obeys the following differential equation

\begin{equation}
\partial_\mu 'U = -ie \, 'B_\mu \, 'U,
\end{equation}

\begin{equation}
= -ieS \, B_\mu \, S^{-1} \, 'U - (\partial_\mu S) \, S^{-1} \, 'U.
\end{equation}
On account of Eq. (20), the same differential equation obeys the matrix $SU$. Since the two matrices $'U$ and $SU$ satisfy also the same boundary condition

$$SU|_{\tau = -\infty} = 1, \quad 'U|_{\tau = -\infty} = 1$$

it follows that they are identical

$$'U = SU, \quad 'U^+ = U^+ S^{-1}.$$  

After substituting (31) into (27) we obtain

$$'\mathcal{F}_{\mu \nu} = \mathcal{F}_{\mu \nu}.$$  

5. The matrix fields $F_{\mu \nu}$ can be replaced by three real (hermitian) fields $\mathcal{F}_{\mu \nu}^i$

$$\mathcal{F}_{\mu \nu}^i = 1/2 \text{Tr} (\mathcal{F}_{\mu \nu} \tau^i).$$

The formulas relating directly $\mathcal{F}_{\mu \nu}^i$ to isovectors $f_{\mu \nu}$ are, however, rather complicated.

The simplicity of Eqs. (24) and (25) as compared to Eqs. (10) and (11) suggests that gauge-independent fields $\mathcal{F}_{\mu \nu}$ may prove helpful in quantizing the Yang—Mills theory. Another interesting problem is an analogous construction of coordinate-independent variables in general theory of relativity.

**Summary.** The gauge-independent field variables in the Yang-Mills theory are explicitly constructed. The field equations obeyed by these fields are linear and coincide with field equations in electrodynamics.

**REFERENCES**