Exact Solutions of Nonrelativistic Classical and Quantum Field Theory with Harmonic Forces

To the memory of Marek Kac and Stanislaw Marcin Ulam – great scientists and our fine friends

IWO BIALYNICKI-BIRULA
Institute for Theoretical Physics, Polish Academy of Sciences, Lotnikow 32/46, 02-668 Warsaw, Poland

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Abstract. Nonlinear field equations describing nonrelativistic particles interacting via harmonic forces are exactly solved. Collective variables of the system are identified and their evolution is found. Full quantum solution of the initial value problem is exhibited and compared with the classical solution of the same problem.

Harmonic oscillators have frequently served as the best laboratory for theoretical physicists. Using harmonic oscillators one may construct models at all levels of complexity: from a single classical oscillator in one dimension to relativistic quantum fields.

In this Letter I shall use the field-theoretic description of a system of harmonic oscillators to study the relationship between the quantum and the classical theories. This relationship is very simple in the standard approach to the $N$-oscillator problem, since the classical solution of the equations of motion can also be directly used in the quantum case, due to the linearity of the equations. The linearity of the equations of motion, however, is completely lost in the formulation of the many-oscillator problem in terms of the annihilation and creation operators.

The equation of motion for the field operator $\Psi(x, t)$ annihilating an oscillator at the point $x$ has the form

$$ih \frac{\partial}{\partial t}\Psi(x, t) = \left[ \frac{p^2}{2m} + m\omega^2 + \frac{2}{2m} \int dx' \Psi^*(x', t)\Psi(x', t)(x' - x)^2 \right] \Psi(x, t),$$

(1)

where

$$p = -ih \frac{\partial}{\partial x},$$

(2)

and it has the Hermitian conjugate form for the field operator $\Psi^*(x, t)$ creating an oscillator at $x$. The nonlinearity of these equations is due to a greater generality of the description as compared to the standard approach; the field equations fully describe the system made of an unspecified number of oscillators. This nonlinearity is a source of interesting differences between the classical and quantum descriptions.
In order to simplify the formulas I shall choose the units of length and time that are built from the three constants appearing in the problem. This choice results in the replacements

\[
x \rightarrow (\hbar/m \omega)^{1/2} x, \quad t \rightarrow t/\omega, \quad \Psi \rightarrow (\hbar/m \omega)^{1/4} \Psi
\]

and the three constants drop out of all the equations.

The field equation (1) and the total Hamiltonian of the system in the dimensionless form are:

\[
i \partial_t \Psi(x, t) = \left[\frac{p^2}{2} + N(x - X(t))N^2/2 + U(t)\right] \Psi(x, t),
\]

\[
H = \frac{p^2}{2N} + T(t) + NU(t),
\]

where I have introduced the following set of operators representing the global, collective variables

\[
N = \int dx \Psi^\dagger \Psi,
\]

\[
X(t) = \int dx \Psi^\dagger x \Psi,
\]

\[
T(t) = \frac{1}{2} \int dx \left(p^2 \Psi^\dagger \Psi \right) - \frac{1}{2N} P^2,
\]

\[
U(t) = \frac{1}{2} \int dx \Psi^\dagger x^2 \Psi - \frac{1}{2N} X^2(t),
\]

\[
W(t) = \frac{1}{4} \int dx \left(\Psi^\dagger (xp + px) \Psi \right) - \frac{1}{4N} \left( (X(t)P + PX(t)) \right).
\]

These operators will prove very useful in the forthcoming analysis. They form a closed commutator algebra, because their only nonvanishing equal-time commutators are

\[
[X, P] = iN, \quad [T, U] = -2iW,
\]

\[
\]

The fact that the $X$ and $P$ operators commute with the remaining global operators indicates that the center of mass motion has been completely separated from the quadrupole oscillations described by the operators $T$, $U$ and $W$. These last three operators obey the commutation relations of the generators of the Lorentz group in 2 + 1
dimensions, with the following correspondence between them

\[ T + U \rightarrow L_{12}, \quad T - U \rightarrow L_{02}, \quad W \rightarrow L_{01}. \] (13)

The same group appears as the dynamical group for a single harmonic oscillator [1].

As I have already indicated in the formulas (5)-(11), the operators of the number of particles \( N \), of the total momentum \( P \), and of the total energy \( H \) do not depend on time. The time-dependence of the remaining global operators may be determined by integrating the following set of linear equations of motion that results from taking the commutators with the Hamiltonian \( H \),

\[
\begin{align*}
dX(t)/dt &= P, \quad dT(t)/dt = -2NW(t), \\
dU(t)/dt &= 2W(t), \\
dW(t)/dt &= T(t) - UU(t).
\end{align*}
\] (14a)

The solution is

\[
\begin{align*}
X(t) &= Pt + X, \\
T(t) &= T \cos^2 \Omega t + NU \sin^2 \Omega t - 2Wt \sin \Omega t \cos \Omega t, \\
U(t) &= U \cos^2 \Omega t + TN^{-1} \sin^2 \Omega t + 2W^{-1} \sin \Omega t \cos \Omega t, \\
W(t) &= W(\cos^2 \Omega t - \sin^2 \Omega t) + (T - NU)\Omega^{-1} \sin \Omega t \cos \Omega t.
\end{align*}
\] (15a-c)

where \( \Omega = \sqrt{N} \) and all the operators on the right-hand side of these equations are taken at \( t = 0 \).

The time-dependence of the global variables can be fully determined prior to solving the field equations. Also, the time-dependence is exactly the same in both the classical and quantum cases. It is not so for the field operators.

In order to solve the field equation, I shall first make the substitution

\[
\Psi(x, t) = \exp[iNH] \Psi_1(x, t)
\] (16)

to get rid of the \( t \)-dependence of the \( X(t) \) and \( U(t) \) operators in Equation (4). In the resulting equation for \( \Psi_1 \),

\[
i \partial_t \Psi_1(x, t) = \left[ p^2/2N + T + (N + 1)U + p^2/2 + N(x - X/N)^2/2 \right] \Psi_1(x, t),
\] (17)

the operator in the square bracket clearly plays the role of the effective (time-independent) Hamiltonian for the \( \Psi_1 \) field. This operator is made up of three groups of terms. The first term,

\[
K = p^2/2N,
\] (18)

describes the total kinetic energy of the system of \( N \) oscillators. The next two terms describe the energy of the collective quadrupole oscillations of the system,

\[
H_Q = T + (N + 1)U.
\] (19)
Finally, the last two terms describe the Hamiltonian of the relative motion for an individual oscillator, moving under the influence of all the oscillators with respect to the center of mass. The collective oscillations and the motion of an individual oscillator are coupled through the appearance of the center of mass coordinate $X/N$ in the potential energy term. This coupling may, however, be easily removed by a unitary transformation: a suitable rotation in the $(x/\sqrt{N}, X)$-plane and the $(P, p/\sqrt{N})$-plane generated by the unitary operator $S$,

$$
S = \exp \left[ i2(xP - pX)/\sqrt{N} \right],
$$

(20)

$$
S^t XS = x \cos \lambda + (X/\sqrt{N}) \sin \lambda,
$$

(21a)

$$
S^t XS = -x \sqrt{N} \sin \lambda + X \cos \lambda,
$$

(21b)

$$
S^t pS = p \cos \lambda + (P/\sqrt{N}) \sin \lambda,
$$

$$
S^t PS = -p \sqrt{N} \sin \lambda + P \cos \lambda.
$$

When $\lambda$ is set equal to $\lambda_0$,

$$
\lambda_0 = \arctan (1/\sqrt{N}),
$$

(22)

the two motions of the system become uncoupled. The field operator $\Psi_2$,

$$
\Psi_2(x, t) = S\Psi_2(x, t)
$$

(23)

obeys the field equation

$$
i \partial_t \Psi_2(x, t) = [K + H_G + H_S] \Psi_2(x, t),
$$

(24)

where

$$
H_S = \frac{1}{2}(p^2 + (N + 1)x^2).
$$

(25)

is the Hamiltonian of a single oscillator.

The free motion of the system as a whole and the collective oscillations do not affect the $x$-dependence of the field operator. The terms corresponding to these motions may be removed from the field equation (24) by the following transformation

$$
\Psi_2(x, t) = \exp \left[ -it(K + H_G) \right] \Psi_3(x, t).
$$

(26)

The field operator $\Psi_3$ obeys a single-particle Schrödinger equation with the oscillator strength multiplied by $N + 1$,

$$
i \partial_t \Psi_3(x, t) = H_S \Psi_3(x, t).
$$

(27)

The Cauchy problem for this linear equation can be solved by standard methods, for example, by expanding the field operator at $t = 0$ into the eigenfunctions of the harmonic oscillator. I shall write this solution in the following symbolic form

$$
\Psi_3(x, t) = \exp \left[ -it(p^2 + (N + 1)x^2)/2 \right] \Psi_3(x, 0).
$$

(28)
I shall combine now all the intermediate steps in order to obtain the final formula,

\[ \Psi(x, t) = \exp\left[i(T + NU)\right] \exp\left[-i(T + (N + 1)U)\right] \times \]
\[ \times \exp\left[i\lambda_0(xP - pX(t))/\sqrt{N}\right] \exp\left[-i(p^2 + (N + 1)x^2)/2\right] \times \]
\[ \times \exp\left[-i\lambda_0(xP - pX)/\sqrt{N}\right] \Psi(x, 0). \] (29)

This is the complete solution of the initial value problem for the quantum field operator. I shall compare it now with the solution of the Cauchy problem for the same equation (1) that can be obtained along similar lines. The main difference is that \(X(t), U(t)\) and \(N\) in Equation (4) are now c-numbers, which somewhat simplifies the derivation. The solution for the classical field reads:

\[ \psi_c(x, t) = \exp\left[-i \int_0^t dt U(t)\right] \exp\left[i(xP - pX(t))/N\right] \times \]
\[ \times \exp\left[-i(p^2 + Nx^2)/2\right] \psi_c(x, 0). \] (30)

In order to exhibit the similarity between the classical and the quantum solutions, I shall apply several simple transformations to bring expression (29) to the form

\[ \Psi_q(x, t) = T \exp\left[-i \int_0^t dt U(t)\right] \exp\left[i\lambda_0(xP - pX(t))/\sqrt{N}\right] \times \]
\[ \times \exp\left[-i(p^2 + (N + 1)x^2)/2\right] \exp\left[-i\lambda_0(xP - pX)/\sqrt{N}\right] \Psi_q(x, 0). \] (31)

Every part of this expression has its direct classical counterpart. One obtains the classical formula from the quantum formula if one takes the limit of large \(N\), when

\[ \arctan(1/\sqrt{N})/\sqrt{N} \sim 1/N, \quad N + 1 \sim N \] (32)

and if one disregards the noncommutativity of the \(U(t)\) operators at different times. The reconstruction of the quantum field at the time \(t\) from the knowledge of the classical solution of the initial value problem is practically impossible. For example, how would one know that \(1/\sqrt{N}\) is to be replaced by \(\arctan(1/\sqrt{N})\)?

All the derivations in this Letter were given for one-dimensional oscillators, but the solution in any number of dimensions may be found along similar lines.

The motivation for this work was simply to learn whether the explicit solution can be found, because if something can be done then it can be done for harmonic oscillators. As to the usefulness of my results, I have no opinion at all. Perhaps someone else could see whether they are good for anything.

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References