Evolution modes of the vacuum Wigner function in strong-field QED

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The physical content of a non-covariant quantum-field theoretic transport formalism for strong-field QED is addressed within an integral formulation of the evolution equations which highlights the relation to acoustic and Zitterbewegung modes of the field-free limit. Through the computation within this framework of the electric susceptibility of the vacuum at the one-loop level, it is established that the off-shell behaviour of the electron-positron field is correctly incorporated. The Zitterbewegung mode is seen to be solely responsible at the one-loop level for vacuum polarization. Implications for the full renormalisation of the theory are taken up.

1. Introduction

Among the attempts to formulate relativistic quantum field theories in a manner useful for the non-perturbative treatment of transport and non-equilibrium phenomena [1–3], a promising option is the equal-time approach introduced in [4] for problems in Quantum Electrodynamics (QED) involving strong classical electrodynamic fields. Its most apparent advantage is that the dynamical problem can be expressed as a conventional initial value problem, and hence should be amenable to numerical solution. It is not clear that other formulations [1], which insist on Lorentz covariance, admit similar treatment. Given that analytic solutions to QED problems involving classical electromagnetic fields are available only for a limited number of idealised field configurations (namely, the plane-wave electromagnetic field and constant homogeneous fields of infinite extent), there is certainly room for a tractable numerical method for the controlled treatment of more realistic problems involving per force inhomogeneous external fields of finite extent. Independent of their outcome, such QED studies form a logical prelude to more ambitious efforts to grapple with the inclusion of quantum fluctuations in the gauged field and the extension to non-Abelian gauge theories (in particular, Quantum Chromodynamics).

Some confusion surrounds the physical content of the equal-time formalism and its relation to conventional treatments of QED. One broad concern is whether or not the equal-time formalism can accommodate correctly the off-shell dynamics of the electron–positron field. Another more specific issue which is largely unresolved is that of renormalisation and its implementation within the equal-time formalism. In this paper, we consider what light is shed on these concerns by use of an integral equivalent of the evolution equations in the equal-time formalism.

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2. Equal-time strong-field QED: dual matrix realisation

The formalism introduced in [4] addresses the response of the quantum electron–positron vacuum to strong external classical fields. To this end, an explicitly gauge-invariant equal-time Wigner function

\[ W_{\alpha\beta}(r,p,t) = -\frac{1}{2} \int d^3s \ e^{-i\Phi} \exp\left(-ie \int_{-1/2}^{1/2} d\lambda s \cdot A(r + \lambda s, t)\right) \langle \Phi | \left[ \Psi_\alpha(r + \frac{1}{2}s, t), \Psi_\beta^\dagger(r - \frac{1}{2}s, t) \right] | \Phi \rangle \]

(1)

is introduced. Here, |\Phi\rangle denotes the QED vacuum state and \( \Psi^\dagger \) (\( \Psi \)) is the quantum field operator of the electron–positron field (spinorial indices \( \mu, \nu \)). Use of the straight line integral of the vector potential \( A \) to achieve gauge-invariance has the desirable consequence that \( p \) can be interpreted as a physical kinetic momentum and not as a gauge-dependent canonical momentum. Employing the Clifford algebra decomposition

\[ W(r,p,t) = \frac{1}{4} \left( f_0(r,p,t) + \sum_{i=1}^{3} \rho_i f_i(r,p,t) + \sigma \cdot g_0(r,p,t) + \sum_{i=1}^{3} \rho_i \sigma \cdot g_i(r,p,t) \right) \]

(2)

(\( \rho_k \) and \( \sigma_k \) are the 4 x 4 Dirac matrices constructed from all possible twofold external products of the 2 x 2 identity matrix and the Pauli matrices among themselves), one can define sixteen real-valued phase-space distribution functions describing vacuum properties: \( f_k \) and the components of \( g_k, k = 0, \ldots, 4 \). In brief, these components have the following physical significance: linear combinations of \( f_0 \) and \( f_1 \) and \( g_0 \) and \( g_1 \) yield the charge and current distributions, respectively, of left- and right-handed fermions (\( f_3 \) by itself is the mass distribution for all fermions, independent of their helicity); \( g_2 \) and \( g_3 \) are the electric polarisation and magnetisation densities in phase-space, respectively.

The dynamical problem amounts to solution of the Maxwell equations for the classical electric field \( E \) and magnetic induction \( B \) with a charge density \( \rho \) and current density \( j \) constructed from \( W_{\mu\nu} \) via (we ignore for the moment the need for renormalisation)

\[ \rho(r,t) = \rho_{\text{ext}}(r,t) + e \int \frac{d^3p}{(2\pi)^3} f_0(r,p,t), \]

(3)

and

\[ j(r,t) = j_{\text{ext}}(r,t) + e \int \frac{d^3p}{(2\pi)^3} g_0(r,p,t), \]

(4)

while the field equations for \( \Psi \) and \( \Psi^\dagger \) (in the temporal gauge) imply that the components of \( W(r,p,t) \) must satisfy an evolution equation of the Schrödinger-like form (here and elsewhere, inessential functional dependences are suppressed)

\[ i\partial_t W = (L_0 + L_1) W, \]

(5)

where \( W \) denotes a column vector with the sixteen functions \( f_k \) and \( g_k \) as components. Field-free evolution is governed by \( L_0 \); dependence on \( E \) and \( B \) arises through non-local operators

\[ \mathcal{B}^k = e \int_{-1/2}^{1/2} d\lambda (-i\lambda)^k B(r + i\lambda \partial_{p,t}) \times \partial_p, \]

(6)

appearing in \( L_1 \).
With the “dual” ordering
\[ W = \left( \begin{array}{c} W_0 \\ W_1 \\ W_2 \\ W_3 \end{array} \right), \quad W_i = \left( \begin{array}{c} f_i \\ g_i \end{array} \right), \]
(7)
the two hermitian operators \( L_0 \) and \( L_1 \) read
\[ L_0 = -i \left( \begin{array}{cccc} 0 & \mathcal{D} & 0 & 0 \\ 0 & -2m \cdot \mathbf{1}_4 & 0 & \mathcal{D} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad L_1 = -i \left( \begin{array}{cccc} \mathcal{E}^0 \cdot \mathbf{1}_4 & \mathcal{B} & 0 & 0 \\ \mathcal{B} & \mathcal{E}^0 \cdot \mathbf{1}_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{B} \cdot \mathcal{E}^0 \cdot \mathbf{1}_4 \end{array} \right), \]
(8)
where it is understood that all the entries represent 4x4 matrices acting on the \( W_i \)’s (\( \mathbf{1}_4 \) denotes the 4x4 unit matrix). The 4x4 matrix \( \mathcal{D} \) is given by
\[ \mathcal{D} = \left( \begin{array}{cccc} \nabla_x & \nabla_y & \nabla_z \\ \nabla_y & 0 & -2p_z \\ \nabla_z & -2p_y & 0 \end{array} \right) \equiv \{ \nabla, 2p \}; \]
(9)
likewise,
\[ \mathcal{B} = \{ \mathcal{B}^0, 2\mathcal{B}^1 \}. \]
(10)

We term (8) through (10) the dual matrix realisation of the evolution equations for \( W \) because the 4x4 matrices \( \mathcal{D} \) and \( \mathcal{B} \) are related to their unstarred partners through an operation reminiscent of the duality transformation encountered in electromagnetic theory: \( \mathcal{D} \) (\( \mathcal{B} \)) is obtained from \( \mathcal{D} \) (\( \mathcal{B} \)) by the replacement of \( \mathbf{V} \) (\( \mathbf{B} \)) with \( 2p \) (\( 2\mathbf{B} \)) and \( -\mathbf{V} \) (\( -\mathbf{B} \))—i.e.
\[ \mathcal{D} = \{ 2p, -\mathbf{V} \}, \quad \mathcal{B} = \{ 2\mathbf{B}, -\mathbf{B} \}. \]
(11)

An advantage of the dual representation is that the 4x4 sub-matrix \( \mathcal{D} \) and its dual \( \mathcal{D}^* \) commute (in fact, they are proportional to each others inverses): \( \mathcal{D} \mathcal{D}^* = 2p \cdot \mathbf{V} \cdot \mathbf{1}_4 = \mathcal{D} \mathcal{D} \). Treatment of \( L_0 \) thus reduces, in effect, to manipulation of a 4x4 matrix.

3. Integral form of evolution equation and field-free modes

The desired integral form reads
\[ W(r, p, t) = W^0(r, p, t) - i \int d^3r' dt' G_{\text{ret}}^0(r - r', p, t - t') L_1(r', \partial p, t') W(r', p, t'), \]
(12)
where \( W^0 \) is a solution of the field-free evolution problem reproducing the initial condition specified for \( W \) and \( G_{\text{ret}}^0 \) is the associated retarded Green function (we suppose adiabatic switching-on of the fields). By construction, \( G_{\text{ret}}^0 \) must satisfy
\[ (i\partial_t - L_0) G_{\text{ret}}^0(r - r', p, t - t') = i\delta(r - r') \delta(t - t'), \]
subject to the causality condition \( G_{\text{ret}}^0(r, p, t) = 0 \) for \( t < 0 \). Formally,
\[
G^0_{\text{reg}}(r, p, t) = \partial(t) \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} \exp(-iL_0[q] t),
\]
(14)

where \( L_0[q] \) is the hermitian matrix obtained from \( L_0 \) above by replacing \( \nabla \) everywhere by \( iq \). Invoking the spectral representation of \( L_0[q] \),

\[
\exp(-iL_0[q] t) = e^{-i\omega_+ t} P_{\omega_+} + e^{i\omega_+ t} P_{-\omega_+} + e^{-i\omega_- t} P_{\omega_-} + e^{i\omega_- t} P_{-\omega_-},
\]
(15)

where \( \pm \omega_\pm \) are the four distinct (circular) eigenfrequencies of \( L_0[q] \) and \( P_{\pm\omega_\pm} \) are the corresponding eigenspace projectors:

\[
P_{\pm\omega_\pm} = \frac{1}{2\omega_\pm} \left( L_0[q] + \pm \omega_\pm \right)(L_0[q] - \omega^2_\pm).
\]
(16)

In terms of energies \( E_{p\pm q/2} \) (\( E_p = \sqrt{m^2 + p^2} \)), the eigenfrequencies are

\[
\omega_\pm = \frac{1}{\hbar} \left( E_{p+q/2} \pm E_{p-q/2} \right).
\]
(17)

The integral equation for Wigner functions \( W(r, p) \) appropriate to time-independent problems and appealed to in [4] is obtained from (12) under the assumption of adiabatic switching-on of static external fields \( E \) and \( B \).

An appealing aspect of the integral form of the evolution equations is that it makes explicit the time dependence of the normal modes of field-free evolution at the level of the dynamics in phase-space. From the time dependence of our result for the evolution operator \( \exp(-iL_0[q] t) \), we read off that these phase-space modes have circular frequencies \( \pm \omega_\pm \). As in the limit of long wavelength disturbances of the vacuum \( \langle q \rangle \rightarrow 0 \), \( \omega_- \) is proportional to \( q \) — specifically,

\[
\omega_- \rightarrow v \cdot q,
\]
(18)

where \( v \) is the relativistic velocity \( v = p/E_p \), we may identify the modes with frequency of \( |\omega_-\rangle \) as acoustic. The remaining modes are always of high frequency (observe that \( \hbar \omega_+ > 2E_p \)). Their origin can be traced back, on the one hand, to that of the Zitterbewegung found within relativistic quantum mechanics; on the other hand, from the perspective of response theory, these modes resemble the optical modes of solid-state plasmas in as much as \( \hbar \omega_+ \) tends to a \( q \)-independent constant (a “mass gap” > 2\( m \)) in the limit \( q \rightarrow 0 \). Within Dirac’s construction for the electron–positron vacuum, such behaviour has a natural explanation in terms of excitation of electrons from the Dirac sea (the obvious analogue of optical exciton formation). The implication is that Zitterbewegung modes are intimately related to the mechanism of pair-production. This interpretation of the significance of Zitterbewegung modes will be taken up in the next subsection.

With reference to these normal modes of the field-free evolution, one qualitative distinction between the character of homogeneous and inhomogeneous external field problems can be drawn. In inhomogeneous external field problems, we have to deal with two continua of acoustic and Zitterbewegung modes, which, for sufficiently large values of \( q \), overlap; for homogeneous problems \( \langle q \rangle = 0 \), there is only a continuum of Zitterbewegung modes which never overlaps with the acoustic branch (for which \( \omega_- \equiv 0 \)). As a consequence, for inhomogeneous field problems, we anticipate instabilities in the phase-space dynamics which parallel those found in plasmas when the acoustic and optical mode-branches overlap. Furthermore, these instabilities would be completely foreign to the homogeneous field problems studied to date within the equal-time formalism.

Of more use to us than (12) is the integral equation for the Fourier transform (throughout, \( \tilde{f} \) will denote the Fourier transform of \( f \))

\[
\tilde{W}(q, p, \omega) = \int dt \int d^3r \exp(i\omega t - iq \cdot r) W(r, p, t)
\]
(19)
which reads
\[ \tilde{W}(q,p,\omega) = \tilde{W}_0(q,p,\omega) - i \tilde{G}_{\text{ret}}(q,p,\omega) \int \frac{d\omega'}{2\pi} \int \frac{d^3q'}{(2\pi)^3} \tilde{L}_1(q-q',\omega-\omega') \tilde{W}(q',p,\omega'). \] (20)

Provided the fields \( E \) and \( B \) are of finite spatial extent, \( \tilde{E}^0 \) and \( \tilde{B}^k \) in \( \tilde{L}_1 \) can be written in terms of a momentum-shift operator
\[ s_k[q \cdot \partial_p] = \int \frac{d\lambda}{-\sqrt{2}} (-i2)^k \exp(-i\lambda \cdot q) \]
(21)
as
\[ \tilde{E}^0(q,\omega) = e[\tilde{E}(q,\omega) \cdot \partial_p]s_0[q \cdot \partial_p], \quad \tilde{B}^k(q,\omega) = e[\tilde{B}(q,\omega) \times \partial_p]s_k[q \cdot \partial_p]. \] (22)

Using (16), the expression for \( \tilde{G}_{\text{ret}}^0 \) implied by (14) and (15) can be identified as the pole decomposition of
\[ -i \tilde{G}_{\text{ret}}^0 = \frac{(\omega + L_0[q] \omega^2 - (\omega^*_+ + \omega^2 - L_0^2[q]))}{[(\omega^*_+)^2 - \omega^*_+ \omega^2] [(\omega^*_+)^2 - \omega^2]} \] (23)
The causality condition on \( \tilde{G}_{\text{ret}}^0 \) is guaranteed by the standard device of prescribing pole positions via introducing \( \omega^* = \omega + i\epsilon (\epsilon \to 0^+). \)

4. Linear response of the Dirac vacuum: electric susceptibility

To gain some appreciation for the physical content of the integral formulation, in particular, the significance of the Zitterbewegung mode, we consider the response of the Dirac vacuum to weak external fields. This limit has the advantage that it is amenable to analytical treatment. We shall take up the implications of our findings for strong fields below.

We focus on the charge density induced on application of external fields since, among the various physical characteristics described by \( W \), this one is the most obviously related to the process of pair creation. (It is also germane to the discussion of renormalisation within this variant of QED.) We adopt the linear response solution for \( \tilde{W} \), by which we mean \( \tilde{W} - \tilde{W}_0 \) calculated within the Born approximation to the solution of (20), i.e.
\[ (\tilde{W} - \tilde{W}_0)(q,p,\omega) \approx \tilde{W}_0^{(1)}(q,p,\omega) = -i \tilde{G}_{\text{ret}}^0(q,p,\omega) \tilde{L}_1(q,\partial_p,\omega) W^D(p), \] (24)
where, for \( W^0 \), the Wigner function \( W^D(p) \) corresponding to the translationally invariant vacuum state of the non-interacting electron–positron field is used. In the language of Feynman diagrams, we are working to one fermion-loop order.

The Wigner function \( W^D \), which may be computed directly from the definition of \( W \), has non-zero components
\[ f_3^D = -\frac{2m}{E_p}, \quad g_1^D = -\frac{2p}{E_p}. \] (25)
Observe that, consistent with the Dirac picture of the translationally invariant vacuum state of the non-interacting electron–positron field as a filled sea of negative energy electron states (spin-up and spin-down), this result for \( W^D \) implies that the vacuum state is assigned the infinite negative energy
\[ E_{\text{vac}} = -2 \int \frac{d^3p}{(2\pi)^3} E_p. \] (26)
In the calculation of $\tilde{f}_0^{(1)}$, we substitute first explicitly for $G_{ret}^0$ and $L_1$ in eq. (24). Routine evaluation of the resulting matrix product yields

$$i \tilde{f}_0^{(1)}(q, p, \omega) = \frac{\omega^2 q \cdot \vec{E}^0 g^0 D - 4p \cdot q (p \cdot \vec{E}^0 g^0 D + m \vec{E}^0 f_3^D) - 4i\omega q \cdot [p \times (\vec{B}^1 \times g^1 D) + m \vec{B}^1 f_3^D]}{[(\omega^+) - \omega_+^2][(\omega^+) - \omega_2^2]}.$$

(27)

Using eq. (25), the various combinations in the numerator of eq. (27) can be written as

$$q \cdot \vec{E}^0 g^0 D = 2e \int d\lambda \frac{d}{d\lambda} \left[ \exp(-\lambda q \cdot \partial_p) \left( p \cdot \vec{E}/E_p \right) \right],$$

$$p \cdot \vec{E}^0 g^0 D + m \vec{E}^0 f_3^D = 2e \int d\lambda \frac{d}{d\lambda} \left[ \exp(-\lambda q \cdot \partial_p) \left( p \cdot \vec{E}/E_p \right) \right],$$

$$i\omega q \cdot [p \times (\vec{B}^1 \times g^1 D) + m \vec{B}^1 f_3^D]$$

$$= 2e \left( p \cdot q \int d\lambda \exp(-\lambda q \cdot \partial_p) \left( p \cdot \vec{E}/E_p \right) + p \cdot \vec{E} \right) \int d\lambda \frac{d}{d\lambda} E_p - iq),$$

(28)

where, in the last of these combinations, we have invoked the interrelationship $\omega \vec{B} = q \times \vec{E}$ (implied by the Maxwell equations) to substitute $\vec{B}$ by $\vec{E}$. On combining these results together and performing the integrations over $\lambda$, we find that $\tilde{f}_0^{(1)}$ can be reduced to

$$\tilde{f}_0^{(1)}(q, p, \omega) = \frac{e}{E_{p+q/2}E_{p-q/2}} \frac{\omega^+}{(\omega^+)^2 - \omega_+^2} \left( iq \cdot \vec{E} - \frac{4p \cdot q \cdot ip \cdot \vec{E}}{\omega^2} \right),$$

$$\equiv ie^{(0)}(q, p, \omega) \cdot \vec{E}(q, \omega),$$

(29)

where, for subsequent convenience, we have introduced a vector-valued response function (the superscript 0 serves as a reminder that it is independent of field strengths).

A noteworthy feature of (29) is the presence of the Zitterbewegung poles and the absence of the acoustic poles found in $G_{ret}^0(q, p, \omega)$. Since $f_0$ is, in this context, the phase-space distribution of the net charge induced by the application of an external electric field, which, with our choice of initial state requires pair production, this finding amounts to circumstantial evidence that the Zitterbewegung modes are intimately related to a physical process, namely pair production.

As one would expect intuitively, the appearance of a non-zero net induced charge is a distinctive consequence of inhomogeneity in the external electric field. Formally, the Fourier transform of the charge density induced in lowest-order Born approximation by the application of an electric field is related to $f_0(q, p, \omega)$ by

$$\tilde{\rho}_{ind}^{(1)} = \frac{e}{2\pi^3} \int d^3p f_0^{(1)}(q, p, \omega) = ie^2 \int d^3p (2\pi)^3 R^{(0)}(q, p, \omega) \cdot \vec{E}(q, \omega).$$

(30)

By elementary symmetry considerations, only the component of $R^{(0)}$ parallel to $q$ can survive the momentum integration in (30), so that

$$\tilde{\rho}_{ind}^{(1)} = \frac{e^2}{q^2} \left( \int d^3p (2\pi)^3 q \cdot R^{(0)}(q, p, \omega) \right) iq \cdot \vec{E}(q, \omega),$$

(31)

which, in turn, implies that the induced charge density $\tilde{\rho}_{ind}^{(1)}$ vanishes unless $E$ has a non-zero divergence (or $q \cdot \vec{E} \neq 0$).
A unfortunate feature of (31) is that the integral over \( p \) diverges logarithmically \((|\int_0^{p(1)}| \sim p^{-3} \text{ as } p = |p| \to \infty)\). The need for a renormalisation prescription is indicated and it can be based on the observation that the form of (31), specifically the fact that \( \tilde{\rho}_{\text{ind}}^{(1)} \propto \mathbf{q} \cdot \mathbf{E}(\mathbf{q}, \omega) \), allows us to relate the divergent integral to the electric susceptibility of the vacuum to this order, \( \chi^{(0)}(\mathbf{q}, \omega) \): comparing (31) with the general relation in linear response theory for \( \tilde{\rho}_{\text{ind}} \) in terms of electric susceptibility \( \chi \) (for an isotropic medium and our choice of electromagnetic units, \( \rho_{\text{ind}} = -i \mathbf{q} \cdot \mathbf{E} \)),

\[
\chi^{(0)}(\mathbf{q}, \omega) = -\frac{\epsilon^2}{q^2} \int \frac{d^3p}{(2\pi)^3} \mathbf{R}^{(0)}(\mathbf{q}, \mathbf{p}, \omega) \cdot \mathbf{q} .
\]  

In this connection, we note that the proportionality of \( \chi^{(0)}(\mathbf{q}) \) to \( \mathbf{q} \cdot \mathbf{E}(\mathbf{q}, \omega) \) is not a consequence of the specific form of \( \mathbf{R}(\mathbf{q}) \) in (29), but a feature of the linear response approximation. Even for the most general form of \( \mathbf{R}^{(0)} \) admissible by symmetry considerations, namely,

\[
\mathbf{R}^{(0)}(\mathbf{q}, \mathbf{p}, \omega) = R_p \mathbf{p} + R_q \mathbf{q} ,
\]

where \( R_p \) and \( R_q \) are scalar functions of \( \omega \) and the invariants \( p^2, q^2 \) and \( \mathbf{p} \cdot \mathbf{q} \), only components parallel to \( \mathbf{q} \) can survive the momentum integration in (30).

We will find that the renormalised observable susceptibility \( \chi^{(0)}_{\text{obs}}(\mathbf{q}, \omega) \) is fixed by the requirements that it vanish in the limit in which \( \omega \to 0 \) and \( \mathbf{q} \to 0 \). A simple empirical consideration underpins this condition: for weak external fields (of finite extent) which vary arbitrarily slowly in time and space, pair creation effects should vanish in first-order perturbation theory, implying that the observable induced charge density vanishes to this order. In effect,

\[
\chi^{(0)}_{\text{obs}}(\mathbf{q}, \omega) = \frac{\chi^{(0)}(\mathbf{q}, \omega) - \chi^{(0)}(\mathbf{q}, \omega = 0, \omega = 0)}{\epsilon}.
\]

5. Off-shell dynamics in the equal-time formalism: one-loop order

As is well known [5], a similar divergence in the vacuum polarisation induced by a weak external field is encountered within the conventional formulation of perturbative QED and a similar renormalisation procedure is adopted. (The fact that we automatically encounter a logarithmic divergence instead of the quadratic divergence found naively in standard treatments, is a consequence of the manifest gauge invariance of our Wigner function.) Although it is more usual to associate this divergence with charge renormalisation, its implications for the renormalised dielectric function \( \epsilon(\mathbf{q}, \omega) \) of the vacuum (related to \( \chi_{\text{obs}} \) via \( \epsilon(\mathbf{q}, \omega) = 1 + \chi_{\text{obs}}(\mathbf{q}, \omega) \)) have also been considered, the outcome being that [5]

\[
\epsilon(\mathbf{q}, \omega) = 1 + \frac{k^2}{\pi} \int_0^\infty d(\mu^2) \frac{\rho(\mu^2)}{\mu^2[\mu^2 - k^2 - i \text{sgn}(\omega)\epsilon]} ,
\]

where, again, \( \epsilon \) is a positive infinitesimal. To one-loop order, the spectral density

\[
\rho(k^2) = \rho^{(1)}(k^2) = \frac{\alpha}{3} \left(1 + \frac{2m}{k^2}\right) \sqrt{1 - \frac{4m^2}{k^2}} \Theta(k^2 - 4m^2) ,
\]

where \( \alpha \) is the fine structure constant. We shall now proceed to show that, armed with our result for \( \tilde{\rho}_{\text{ind}}^{(1)} \), we recover the standard result for the dielectric function of the vacuum to one-loop order. The calculation will serve to expose the role of the Zitterbewegung mode as well as illustrating that the equal-time formalism of [4] accommodates correctly the off-shell dynamics of the electron–positron field.
A convenient basis for computations involving $f_0^{(1)}$ can be established by making contact with general results in linear response theory through the response function $R^{(0)}$. Viewed as a function of $\omega$, $R^{(0)}$ is analytic in the upper-half complex $\omega$-plane and $|\omega R^{(0)}| \to 0$ as $|\omega| \to \infty$. These properties, which have their origin in the causality of the response to the external fields, imply that $R^{(0)}$ satisfies the Kramers–König-type dispersion relation (ubiquitous in linear response theory)

$$ R^{(0)}(q, p, \omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im}[R^{(0)}(q, p, \omega')]}{\omega' - \omega^+}. $$

(37)

The essential information content of $R^{(0)}$ is thus confined to $\text{Im}[R^{(0)}(q, p, \omega)]$, which, using the identity

$$ \text{Im}\left(\frac{\omega}{(\omega^+)^2 - \omega^+}\right) = -\frac{\pi}{2} \text{sgn}(\omega) \delta(|\omega| - \omega^+), $$

(38)

is given by

$$ \text{Im}[R^{(0)}(q, p, \omega)] = -\text{sgn}(\omega) \frac{\pi}{2E_p + q^2} \frac{1}{E_p - q^2} \delta(|\omega| - \omega^+) \left(q - \frac{q \cdot p}{\omega^2}ight). $$

(39)

Observe that it is only because of the presence of the Zitterbewegung poles that $\text{Im}(R^{(0)})$ is non-vanishing.

For the purpose of the evaluation of integrals over momentum $p$, the argument $|\omega| - \omega^+$ of the delta function in (39) can be treated as a function of $p$, suggesting that, for $\text{Im}(R^{(0)})$ at least, the integration in (30) is benign.

Positive roots of $f(p) = |\omega| - \omega^+_+$ exist provided $k^2 > 4m^2$; they occur at

$$ p = p_0 \equiv \frac{1}{2} |\omega| \sqrt{k^2 - 4m^2} \frac{1}{\sqrt{\omega^2 - (qx)^2}}, $$

(40)

where $x = \cos \theta$, $\theta$ being the angle between $q$ and $p$. On use of (40) in (39), one finds that

$$ \frac{1}{q^2} \text{Im}[R^{(0)}(q, p, \omega)] \cdot q = -\frac{\pi}{\sqrt{(k^2 - 4m^2)^2 - (\omega^2 - (xq)^2)^2}} \left(1 - \frac{x^2}{\omega^2 - (xq)^2}\right) \times \delta(p - p_0) \Theta(k^2 - 4m^2), $$

(41)

which, in turn, implies, integrating first over $p$ in (32) and then performing the now elementary angular integrations, that

$$ \text{Im}[\chi^{(0)}(q, \omega)] = \text{Im}[\chi^{(0)}(q, \omega)] = \text{sgn}(\omega) \rho^{(1)}(k^2). $$

(42)

The linear relation of $\chi^{(0)}_{\text{obs}}$ to $R^{(0)}$ through (32) (only additive contributions are dropped in the renormalisation procedure), together with the fact that $\text{Im}[\chi^{(0)}_{\text{obs}}(q, \omega)]$ is an odd function of $\omega$ (consistent with the real-valuedness of the electric field $E$ and displacement $D$), imply that, starting from the dispersion relation (37) for $R^{(0)}$, we can derive the dispersion relation for $\chi^{(0)}_{\text{obs}}$ in the equal-time formalism

$$ \chi^{(0)}_{\text{obs}}(q, \omega) = 2 \int_{0}^{\infty} \frac{d\omega'}{\pi} \omega' \left(\frac{\text{Im}[\chi^{(0)}_{\text{obs}}(q, \omega')]}{\omega'^2 - (\omega^+)^2} - \frac{\text{Im}[\chi^{(0)}_{\text{obs}}(q = 0, \omega')]}{\omega'^2}ight). $$

(43)

(The oddness of $\text{Im}[\chi^{(0)}_{\text{obs}}(q, \omega)]$ has been used to restrict the range of integration to positive frequencies.) This result facilitates our ultimate goal of comparison with (35). Explicit substitution for $\text{Im}(\chi^{(0)}_{\text{obs}})$ in (43) using (42) along with appropriate changes of integration variables yields the result

$$ \chi^{(0)}_{\text{obs}}(q, \omega) \approx \frac{1}{\pi} \int_{0}^{\infty} d(\mu^2) \rho^{(1)}(\mu^2) \left(\frac{1}{\mu^2 - (\omega^+)^2 - q^2} - \frac{1}{\mu^2}\right).

(44)
where the first and second terms of this last integrand correspond to the first and second terms of the integrand in (43). Obviously, the associated dielectric function $1 + \chi_{\text{obs}}^{(0)}$ coincides at the one-loop level with the standard result (35).

6. Conclusion: implications for strong fields

With the example of the one-loop calculation to guide us, we can formulate a procedure for renormalisation in the presence of external fields of arbitrary strength.

Our renormalisation prescription for the electric susceptibility $\chi$ has an obvious equivalent at the level of the component $\tilde{f}_0^{(1)}$ of the fourier transform of the Wigner function: the Fourier transform of the observable induced charge density (to one-loop order) is recovered with the renormalised phase-space charge density

$$\tilde{f}_0^{(1)} \text{ren} (q, p, \omega) = \tilde{f}_0^{(1)} (q, p, \omega) - e \lim_{q,\omega \to 0} \left[ q \cdot R^{(0)} (q, p, \omega) / q^2 \right] [i q \cdot \tilde{E} (q, \omega)].$$

(45)

In fact, perturbative considerations suggest that, because we work only with classical external electromagnetic fields, renormalisation of $\tilde{f}_0$ at the one-loop level should suffice to all orders – apart from the one-loop diagram already considered, diagrams which could give rise to divergences are absent. Independent confirmation of this assertion is provided by inspection of the $p$-dependence of higher orders of the Born-series solution to (20). Thus, we conjecture that, when we work with external fields of arbitrary strength, the fully-renormalised phase-space charge density is

$$\tilde{f}_0|\text{ren} = \left( \tilde{f}_0 - \tilde{f}_0^{(1)} \right) + \tilde{f}_0^{(1)} \Big|_{\text{ren}} .$$

(46)

Similar perturbative considerations apply, of course, to the other components of $\tilde{W}$. Accordingly, we are led to define the renormalised Wigner function to be

$$\tilde{W}_{\text{ren}} = \left( \tilde{W} - \tilde{W}^{(1)} \right) + \tilde{W}^{(1)} \text{ren} .$$

(47)

Although different physical considerations apply in the renormalisations of components of $\tilde{W}^{(1)}$ other than $\tilde{f}_0^{(1)}$, their determination parallels that of $\tilde{f}_0^{(1)}|_{\text{ren}}$, because many of the details can be formulated in terms of response functions analogous to $R^{(0)}$. The integral equation satisfied by our renormalised Wigner function may be inferred from (20). (In practice, it is also convenient for reasons of numerical stability to eliminate $\tilde{W}^D$ by subtraction.)

In sum, the accomplishment of this paper has been to explicate how the equal-time formalism encapsulates correctly the off-shell dynamics of the electron–positron field at the one-loop level. Despite the simplifications already inherent in the equal-time formalism, full solution of the evolution equations constitutes a formidable problem which the introduction of renormalisation only serves to exacerbate. Our qualitative considerations of this paper indicate that these equations describe a rich phase-space dynamics comprising coupled branches of acoustic and Zitterbewegung modes.

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