Comment on an Exactly Soluble Model of Relativistic Field Theory

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Difficulties connected with the indefinite-energy version of a model of relativistic field theory are pointed out.

Solvable relativistic field-theory models are of great interest, even if they are unrealistic in some respects. Glaser¹ and Białynicki-Birula² proposed such a model in which

\[ \mathcal{L} = \mathcal{L}_0(\phi_+) - \mathcal{L}_0(\phi_-) - \frac{i}{\hbar}[\phi_+ \phi_+ + \phi_- \phi_-] , \]

where

\[ \mathcal{L}_0(\phi) = \frac{1}{2}[(\partial \phi) \partial \phi - m^2\phi^2] , \]

where \( \phi_+ \) are mass-\( m \) neutral scalar fields. Both authors gave the exact classical solution in closed form. This model can be quantized in two different ways: (1) The field \( \phi_- \) creates positive-energy particles, but it is a negative-norm field, or (2) the field \( \phi_- \) is a positive-norm field, but creates negative-energy particles. In either case, \( \phi_+ \) is a normal field; that is, it creates positive-energy particles and is a positive-norm field. Both authors¹,² discussed version (1) and calculated the \( S \) matrix exactly. This \( S \) matrix is pseudo-unitary rather than unitary, and the only nontrivial \( S \)-matrix element is elastic scattering, which occurs only in first order in the coupling constant, \( \lambda \). Białynicki-Birula² also discussed version (2) and gave an explicit expression, quadratic in \( \lambda \), Eq. (17) of Ref. 2, for the two-particle scattering amplitude. Our comment concerns version (2) of the model.

Although the no-particle state \( \lvert 0 \rangle \) of the in fields \( A \) and \( B \), belonging to \( \phi_+ \) and \( \phi_- \), respectively, is an eigenvector of \( P_+ \) [see Eq. (7) of Ref. 2], with eigenvalue zero, \( \lvert 0 \rangle \) is not an eigenvector of the \( S \)-matrix operator [see Eq. (16) of Ref. 2] and cannot be interpreted as the vacuum state of the model. The physical reason for this is that \( \lvert 0 \rangle \) is unstable because of the presence of negative-energy particles, and the reaction \( \lvert 0 \rangle \rightarrow \lvert 2a + 2b \rangle \) is energetically allowed. The states \( A' (k) \lvert 0 \rangle \) and \( B' (k) \lvert 0 \rangle \) are also unstable and are also not eigenstates of \( S \). The condition \( [P_+ , S] = 0 \) is satisfied; however, this condition applied to \( \lvert 0 \rangle \) leads only to

\[ S \lvert 0 \rangle = \omega \lvert 0 \rangle , \quad |\omega| = 1 \]

because of the great degeneracy of states satisfying \( P_+ \lvert \Psi \rangle = 0 \) in a model with negative-energy particles. A similar comment holds for \( \lvert 1 \rangle \).

A proper physical interpretation of this model should start with a vacuum \( \lvert 0 \rangle \rangle \) which is an eigenstate of \( S \). If such a state \( \lvert 0 \rangle \rangle \) exists, it clearly must have continuum normalization and be embedded in a continuum of states having energy-momentum near the origin in momentum space. There should also be a "one-particle" state \( \lvert 1 \rangle \rangle \) which is an eigenstate of \( S \). If such a state exists, it will not have energy-momentum on a discrete mass shell. The proper in and out fields of the model should create \( \lvert 1 \rangle \rangle \) by acting on \( \lvert 0 \rangle \rangle \).

From the mathematical standpoint, because \( \lvert 0 \rangle \rangle \) is the only normalizable eigenstate of \( P_+ \) in Fock space, the failure of (a) implies that \( S \) maps \( \lvert 0 \rangle \) out of Fock space, and, further, the commutation relation

\[ [S, A(x)] = -\int d^n z \Delta (x-z) S(z) \]

implies that \( S \) maps every vector in Fock space out of Fock space. A proper mathematical formulation of this model, if one exists, will require the use of representations of the commutation relations which are not equivalent to the Fock representation.³
Comment on the Absence of Radiative Corrections to the Anomaly of the Axial-Vector Current

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The absence of radiative corrections to the Schwinger-Bell-Jackiw-Adler anomaly of the axial-vector-current Ward-Takahashi identity is demonstrated using normal-product methods.

Adler and Bardeen\textsuperscript{1} have given convincing, but not completely rigorous, arguments that the coefficient \( \gamma \) in the Ward-Takahashi identity of the axial-vector current in (say) quantum electrodynamics (QED),

\[
\delta \langle 0 | T_{\mu \nu}(x)X(0) \rangle = 2iM(0)T_{\mu \nu}(x)X(0) + \gamma(0)T_{\mu \nu}(x)X(0) - \sum_i \delta(x-x_i)A_{\mu \nu}^i + \delta(x-y_i)A_{\mu \nu}^j(0)TX(0),
\]

has no radiative corrections to the second-order (triangle-graph) contribution. Here \( X \) represents any product of the basic fields,

\[
\prod_{i=1}^{N} \phi(x_i) \prod_{j=1}^{N} \bar{\phi}(y_j) \prod_{k=1}^{L} A_{\mu \nu}(z_k).
\]

The principal line of reasoning in Ref. 1 is that the higher-order contributions to \( \gamma \) vanish for a theory in which the photon lines are regulated in a gauge-invariant manner, and that this property should persist when the regulator is removed. The Adler-Bardeen claim has been further supported by a number of explicit computations\textsuperscript{1,2} showing that

\[
\gamma^{(4)} = 0.
\]

A more systematic approach to the problem is provided by normal-product methods in Zimmermann's formulation of renormalized perturbation theory.\textsuperscript{3} As we shall see below, the normal-product algorithm allows one to derive in a straightforward manner certain Callan-Symanzik equations\textsuperscript{4} and Ward-Takahashi identities from which the vanishing of \( \gamma - \gamma^{(2)} \) follows easily. The advantage of this approach is that regulators are completely avoided, the finiteness (and often the gauge invariance) of vertex functions being guaranteed by the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) subtraction scheme.\textsuperscript{3}

In the massive vector-meson model (massive QED) with effective Lagrangian\textsuperscript{5}

\[
\mathcal{L}_{\text{eff}} = (1 + d) \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^\nu \psi - (M - c) \bar{\psi} \psi - \frac{1}{2} (1 - b) \partial_\mu A_\mu \bar{\psi}^\nu A^\nu + \frac{1}{2} (m^2 + a) A_\mu \bar{\psi}^\nu A^\nu + \frac{1}{2} \left( 1 - b \frac{m^2 + a}{m_0^2} \right) \left( \partial_\mu A^\nu \right)^2,
\]

we have shown\textsuperscript{5} that the Schwinger-Bell-Jackiw-Adler anomaly\textsuperscript{6} may be expressed as

\[
(1 + d - s) \delta \langle 0 | T_{\mu \nu}(x)X|0 \rangle = 2(\delta(M - c)|0| T_{\mu \nu}(x)X|0 \rangle + \sum_i \delta(x-x_i)A_{\mu \nu}^i + \delta(x-y_i)A_{\mu \nu}^j(0)TX(0),
\]

\[
+r(0)TN_{(F_{\mu \nu}(x))}X|0 \rangle - \sum_i \delta(x-x_i)A_{\mu \nu}^i + \delta(x-y_i)A_{\mu \nu}^j(0)TX(0),
\]