Properties of linearized Boltzmann equations for fermions

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Schedule

Schedule

- Classical Boltzmann equation
- Linearized Boltzmann equation for fermions
- Estimates for fermionic equation
- Width of the spectral gap
- Difference between classical and quantum case

Assumptions of the Boltzmann equation

Assumptions

- The gas consist of discrete particles
- *Stosszahlansatz* - molecular chaos assumptions
- Single colissions (but not whole equation and its solutions !) are reversable in time

Classical Boltzmann equation

Classical Boltzmann equation

$$Df_1 = B(f_1, f_2, f_3, f_4) \tag{1}$$

D - differential operator, connected with continuity equation in phase space
B - colission integral

Classical Boltzmann equation

Distribution function and conservation rules

$f_i = f(\vec{p}_i)$ $i \in \{1,2,3,4\}$ When

- $\vec{p}_1$ Momentum of the first particle before colission
- $\vec{p}_2$ Momentum of the second particle before colission
- $\vec{p}_3$ Momentum of the first particle after colission
- $\vec{p}_4$ Momentum of the second particle after colission

$$D = \partial_t + \frac{1}{m}\vec{p}_1 \cdot \nabla \qquad (2)$$

$$B = \int d^3p_2 d^3p_3 d^3p_4 \sigma(\vec{p}_1, \vec{p}_2)|\vec{p}_1 - \vec{p}_2|(f_3 f_4 - f_1 f_2)\delta_{\vec{p}}\delta_E \qquad (3)$$

Linearization

$$f_i = F_i + w_i h_i \tag{4}$$

F is a Gaussian function that is an equilibrum solution of the Boltzmann equation. If we want Boltzmann equation to be symmetric, w must be a weight in the scalar product in the Hilbert space that we use $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3, w)$ when the scalar product of f i g is given by formula

$$(f|g) = \int d^3x w(x) f(x) g(x) \tag{5}$$

- To make an linear integral operator from the colission kernel, we neglect non linear parts

Most important properties of the classical Boltzmann equation

Most important properties of the classical Boltzmann equation

- B is a hermitian (symmetric) and non-positive linear operator in Hilbert space $L^2(\mathbb{R}^3, w)$ $(f|Bg)_w = (g|Bf)_w$, $(f|Bf)_w \leq 0$ with the scalar product $(f|g)_w = \int d^3x w(x) f(x) g(x)$
- B is - in the most general case - sum of the compact operator and multiplication by function operator
- B has an 5-dimensional eigenspace at $\lambda = 0$ (combinations of the hydrodynamic modees)
- B generates an semi-group of solutions; solutions of LBE are semigroup
- For such a class of the colission potentials (hard potentials)exist a spectral gap between hydrodynamic and fast-decaying part of the solutions.

Linearization of Boltzmann equation for fermions

- Not the same like classical BE
- Pauli Exclusion Princliple Input $f_i \to f_i(1 - f_{i+_2 mod4})$
- Equilibrum function - Fermi-Dirac F^{FD}
- Linearization formula:
$$f = F^{FD}(1 - F^{FD})(1 + wh) \tag{6}$$
- Factor $G = F^{FD}(1 - F^{FD})$ is estimaded by classical Gaussian functions and that give a way for following estimation:

$$\sqrt{G_1 G_2 G_3 G_4} \geq \frac{(C^{FD})^2}{(1 + C^{FD})^4} e^{-\frac{p_1^2 + p_2^2}{p_T^2}} \tag{7}$$

Estimation from above:

$$\sqrt{G_1 G_2 G_3 G_4} \leq (C^{FD})^2 e^{-\frac{p_1^2 + p_2^2}{p_T^2}} \tag{8}$$

Linearized Boltzmann equation for fermions

$$B^{FD}h(p_1) = \frac{1}{w(p_1)} \int d^3p_2 \int d\omega\sigma(\omega, |p_1 - p_2|)|p_1 - p_2| q_L^{FD}(h) \tag{9}$$

$$q_L^{FD} = \sqrt{G_1 G_2 G_3 G_4}\left(\frac{w_3 h_3}{F_3^{FD}(1 - F_3^{FD})} + \frac{w_4 h_4}{F_4^{FD}(1 - F_4^{FD})} - \frac{w_1 h_1}{F_1^{FD}(1 - F_1^{FD})} - \frac{w_2 h_2}{F_2^{FD}(1 - F_2^{FD})}\right) \tag{10}$$

- for $F^{FD} < 1$ with the same formula for cross-sections LBEF has all properties of the LBE

Structure of LBE

Division on compact and continous part

For classical or fermionic B when $w = F^{\alpha}$ for - for example $\alpha = 1$ there are following statement:

$$B = K - \nu \tag{11}$$

when K is an compact operator (disturbing spectrum only on discrete fragment due to M.Reed and M.Simon) andν is multiplication by function operator. ν for classical LBE is given by following formula;

$$\nu(p_1) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} d^3 p_2 d\omega \sigma(|p_1 - p_2|, \omega) e^{-\left(\frac{p_1^2}{p_T^2}\right)} \tag{12}$$

ν for fermions in estimated from above and below by that formula, with non-zero, finite constants.

- That give us method for spectral gap estimation - spectrum of B or fermionic B is on the left of beginning of continous of discrete spectrum

Spectral gap in classical Boltzmann equation

- C.Baranger C.Mouhot (2003) - assumptions form below for width of the spectral gap for hard potentials in classical equation

$$S_\gamma \geq \frac{\pi(\gamma/8)^{\gamma/2} e^{-\gamma/2}}{24} \qquad (13)$$

We can make estimation for fermionic spectral gap from estimations for fermionic equations

Spectral gap for the contact potential in Born Approcimation

- Contact potential in Born Approcimation - $\gamma = 1$
- C.Baranger C.Mouhot (2003) - estimation from below for spectral gap S_γ
- From estimations 7 and Baranger and Mouhot's estimations we have following:

$$S_\gamma \geq \frac{1}{(1+C^{FD})^5}\left(\frac{C^{FD}}{C}\right)^3 \frac{\pi e^{-1/2}}{48\sqrt{2}} \tag{14}$$

When $C^{FD} = e^{\frac{\mu}{k_B T}}$ - μ is a chemical potential for fermionic gas, T is the temperature of the equilibrum state and $C = \frac{nA^d}{(2\pi m k_B T)^{d/2}}$

Hard and soft potential in classical and quantum cases

Definition

- Hard - when $\exists\ \nu_o > 0\ \forall p\ \nu(p) \geq \nu_0$
- Soft - when $\nu(p)$ may tends to 0 when $p \to \infty$

Examples

- Classical mechanics - power potentials $\Phi(r) = \frac{A}{r^n}$ for $n \geq 4$ are hard, e.g $n = 6$ in dipol-stimulated dipol Van der Waals interactions; it means that we have separation between hydrodynamic and exponential-decaying part.
- Born $\Phi(r) = g\delta(r)$ is hard but power ones are soft for any n (!)

Softness of power potential is a contradiction, because give another hydrodynamical behaviour for 'quantum' and 'classical' gases without correspondention.

Hard power potentials in 'quantum' equation

- quasi-Classical WKB approximation (accurate for huge values of momenta) give us an classical formula for cross-section
- Born Approximation estimates total cross-section form below (because is projection only for one dimension of solutions)
- When we make an cut off in p $\nu(p)$ expect the condition for hardness (integral $\int d^3p_2\sigma_{Born}(p_1-p_2)|p_1-p_2|e^{-(\frac{p_2}{p_T})^2}$ is positive for any finite p)
- Connection of Born and quasi-classical part gives us hard power potentials
- Power potentials are hard for any $n \geq 4$ when we do this input

Hydrodynamic sense of the spectral gap

- We can separate any solution of LBE/LBEF for hydrodynamics and ortogonal part to the hydrodynamic modes

$$h = h_{||} + h_{\perp} \tag{15}$$

when $h_{||} = \sum_{i=1}^{5} (h|\psi_i)\, \psi_i$ - hydrodynamical modes and $h_{\perp} = h - h_{||}$

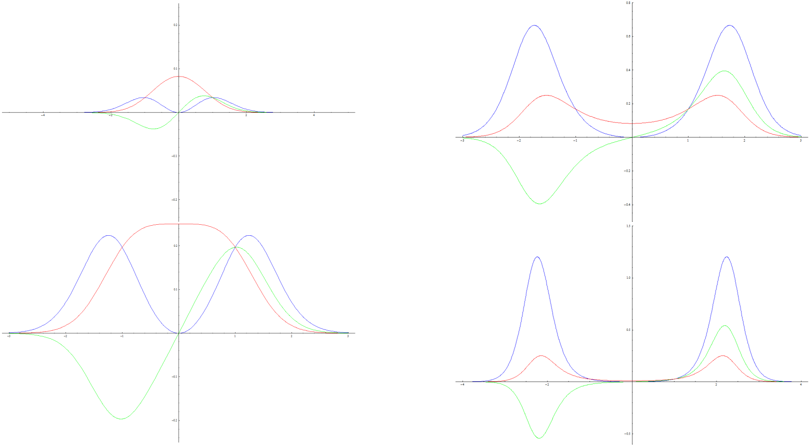
- Due to Hille-Yosida theorem, B and B^{FD} and when we make a Fourier Transform - $B - i\vec{k} \cdot \vec{p}$ o generates a semigroup of solutions. When we exclude hydrodynamic eigenspaces from space of solutions, if there exist a non-zero spectral gap γ, the evolution of $h_{\perp}$

$$h_{\perp}(t) = U_{\perp t} h_{\perp}(t) \tag{16}$$

has estimation $||U_{\perp t}|| \leq e^{-\gamma t}$

- That estimation make us sure that solutions are analitical and in long times aprroximation by solution of system of PDE is accurate

Hydrodynamic modes for different chemical potentials temperatures



The hydrodynamic modes for different chemical potentials $e^{\frac{\mu}{k_B T}}$ 0.1, 1, 100, 10^8

Hilbert expansion and hydrodynamics

- B may be classical or fermionic Boltzmann operator
- We exclude the hydrodynamic subspace from $\mathcal{H}$ and obtain a non-hydrodynamic part of B
- Fourier transform of full LBE $\Rightarrow i\vec{k} \cdot \frac{\vec{p}}{m}$ is a hydrodynamic operator
- We can write a series

$$f = f^{(0)} + \frac{1}{\tau} f^{(1)} + \frac{1}{\tau^2} f^{(2)} + \frac{1}{\tau^3} f^{(3)} + \dots \tag{17}$$

When we put it into BE, and assume that $Bf^{(1)} = \frac{1}{\tau} f^{(1)}$. From the BE $Df = Bf$ we obtain $f^{(1)} = D\frac{1}{\tau} f^{(0)}$

Hilbert expansion and hydrodynamics

Estimation from non-hydrodynamical part

- We divide h onto hydrodynamic and non-hydrodynamic part $h_{||} = \sum_{i=1}^{5} (h|\psi_i)$ ψ_i - hydrodynamical modes and $h_{\perp} = h - h_{||}$
- Due to Hille-Yosida theorem, B and B^{FD} and when we make a Fourier Transform - $B - i\vec{k} \cdot \vec{p}$ o generates a semigroup of solutions. When we exclude hydrodynamic eigenspaces from space of solutions, if there exist a non-zero spectral gap γ, the evolution of $h_{\perp}$

$$h_{\perp}(t) = U_{\perp t} h_{\perp}(t) \tag{18}$$

has following estimation

$$||U_{\perp t}|| \leq e^{-S_{\gamma} t} \tag{19}$$

- We know S_{γ} form Baranger and Mouhot work for classical equation and for fermions - from estimation (7)
- Estimation (19) make us sure that only hydrodynamics has a long-time character

- From non-linear BE we obtained full hydrodynamics; from LBE - linearized hydrodynamics
- Five conservation rules - mass, momentum (three dimensions) and energy
- Stress tensor and heat flux are the inputs from Boltzmann equations
- We compute *kth* component of stress tensor and heat flux from perturbation series
- zeroth order gives us Euler hydrodynamics, first - Navier-Stokes

Stress tensor

$$\Pi_{ij}^{(k)} = \int d^3p \frac{1}{m^2} p_i p_j f^{((k))} \tag{20}$$

in zeroth order - only diagonal terms (pressure) in first order - we have Newtonian viscosity and first derivative of velocity in non diagonal

$$\Pi_{ij}^{(1)} = \eta \partial_i v_j \tag{21}$$

For fermions - viscosity is not equal to viscosity extracted form classical equations, but from (7) for any chemical potential we have

$$\frac{1}{4}\eta_{Classical} \leq \eta_{Fermionic} \leq \eta_{Classical} \tag{22}$$

Summary

- For non-zero temperatures we have good estimations for fermionic LBE
- For power potentials A/r^n we have non-zero spectral gap for $n \geq 4$
- We have separation between fast-decaying part and hydrodynamics
- Spectral gap let us approximate LBE solutions as solutions of PDE system (linearized hydrodynamic equations)
- Non-relativistic Fermionic hydrodynamics has the same equations but with another coefficients

Thank you for attention

Bibliography - papers directly connected with the topic

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