The beauty of the Riemann-Silberstein vector

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Ludwik Silberstein

- Ludwik Silberstein (1872-1948), Polish theoretical physicist, emigrated from Poland to Italy, England, Canada, and finally to the United States
- On November 6, 1919 the results of the Brazilian expedition to measure the deflection of light were reported at the joint meeting of the Royal Society and the Royal Astronomical Society. Eddington recalled that, as the meeting was dispersing, Ludwig Silberstein came up to him and said: "Professor Eddington, you must be one of three persons in the world who understands general relativity." On Eddington demurring to this statement, Silberstein responded, "Don't be modest Eddington." And the Eddington's reply was, "On the contrary, I am trying to think who the third person is!"

Complex form of Maxwell equations

- Almost 100 years ago Ludwik Silberstein published a paper under the title *Elektromagnetische Grundgleichungen in bivectorieller Behandlung* (Annalen der Physik 22, 579 (1907))
- In this paper he introduced a complex vector

$$\boldsymbol{F} = \sqrt{\epsilon_0/2}(\boldsymbol{E} + ic\boldsymbol{B})$$

The RS vector (named by me the Riemann-Silberstein vector) satisfies the equation

$$i\partial_t \boldsymbol{F}(\boldsymbol{r}, t) = c\nabla \times \boldsymbol{F}(\boldsymbol{r}, t)$$

- The energy density and the Poynting vector are bilinear expressions in $\boldsymbol{F}$: $\mathcal{E} = \boldsymbol{F}^* \cdot \boldsymbol{F}$ and $\mathcal{P} = -ic\boldsymbol{F}^* \times \boldsymbol{F}$

The RS vector is a very useful tool

- Take a circularly polarized plane wave

$$\boldsymbol{F}(\boldsymbol{r},t) = Ae^{-i\omega t+ikz}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})$$

- Is this not exceedingly simple?
- The total energy of the field is the norm of $\boldsymbol{F}$

$$E = ||\boldsymbol{F}||^2 = \int d^3r\, \boldsymbol{F}^* \cdot \boldsymbol{F}$$

- The total momentum and the angular momentum of the field also look nice

$$\boldsymbol{P} = -i\int d^3r\; \boldsymbol{F}^* \times \boldsymbol{F} \quad \boldsymbol{M} = -i\int d^3r\; \boldsymbol{r} \times (\boldsymbol{F}^* \times \boldsymbol{F})$$

Complexification saves space and time

- Complex solutions of the Maxwell equations are very often employed as a technical trick
  - Normally, however, only the real part is used and the imaginary part is discarded
  - In contrast, in the case of the RS vector both parts have a physical significance
- Fourier analysis of complex functions is simpler since there are no constrains are that guarantee reality
- In particular we have

$$\int d^3r\, \boldsymbol{F}_1^*(\boldsymbol{r})\boldsymbol{F}_2(\boldsymbol{r}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{F}_1^*(\boldsymbol{k})\tilde{F}_2(\boldsymbol{k})$$

Where is the RS vector found in Nature?

- The Riemann-Silberstein vector is not only a useful mathematical construct that simplifies the analysis
- It also defines the force acting on the magnetic moment of the electron in an EM field
- In order to separate the forces acting on the charge from the spin-dependent forces acting on the magnetic moment, one may take the square of the Dirac operator
- The resulting equation splits into two equations for two-component spinors $\psi = (\psi_1, \psi_2)$ and $\chi = (\chi^{\dot{1}}, \chi^{\dot{2}})$

$$\left(\frac{1}{c^2}(\hbar\partial_t - ieA_0)^2 - (\hbar\nabla + ie\boldsymbol{A})^2 + (mc)^2\right)\psi$$

$$-ie\hbar\,\boldsymbol{\sigma}\cdot(\boldsymbol{E}/c + i\boldsymbol{B})\psi = 0$$

The square of $\boldsymbol{F}$

- The square of $\boldsymbol{F}(\boldsymbol{r}, t)$ is a sum of the two invariants

$$\boldsymbol{F}^2 = \boldsymbol{F}\cdot\boldsymbol{F} = \epsilon_0\left((\boldsymbol{E}^2 - c^2\boldsymbol{B}^2)/2 + ic\boldsymbol{E}\cdot\boldsymbol{B}\right)$$

- All Lorentz transformations acting on the RS vector are represented by complex orthogonal matrices
- The square of $\boldsymbol{F}$ may serve as a substitute for the wave function in wave mechanics. With its use we may define a kind of “velocity” of the electromagnetic field

$$v_\mu = \frac{\boldsymbol{F}^{*2}\partial_\mu\boldsymbol{F}^2 - \boldsymbol{F}^2\partial_\mu\boldsymbol{F}^{*2}}{4i\boldsymbol{F}^{*2}\boldsymbol{F}^2}$$

- Exactly like in wave mechanics, the circulation of v_μ is localized on vortex lines and is quantized $\oint dx^\mu v_\mu = 2\pi n$

The complex Hertz potential

- Very useful representation

$$F(r,t) = \nabla\times\left(\frac{i}{c}\partial_t Z(r,t) + \nabla\times Z(r,t)\right)$$

$$\left((1/c^2)\partial_t^2 - \Delta\right) Z(r,t) = 0$$

- The choice $Z = (1, i, 0)\chi$ is best suited for beams

$$F_x = \left(i\partial_y(\partial_x + i\partial_y) + \partial_z(\frac{1}{c}\partial_t - \partial_z)\right)\chi$$

$$F_y = \left(-i\partial_x(\partial_x + i\partial_y) + i\partial_z(\frac{1}{c}\partial_t - \partial_z)\right)\chi$$

$$F_z = -(\partial_x + i\partial_y)(\frac{1}{c}\partial_t - \partial_z)\chi$$

Expansion into plane waves

- Every solution of the Maxwell equations for $\boldsymbol{F}$ can be expressed in the form

$$\boldsymbol{F}(\boldsymbol{r},t) = \int d^3k\, \boldsymbol{e}(\boldsymbol{k}) \left(\alpha_+(\boldsymbol{k}) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} + \alpha_-(\boldsymbol{k}) e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right)$$

- The complex unit polarization vector $\boldsymbol{e}$ is

$$\boldsymbol{e}(\boldsymbol{k}) = N(\boldsymbol{k})\,(-ik_y\kappa + k_z k_+, ik_x\kappa + ik_z k_+, -\kappa k_z)$$

$$\kappa = k_x + ik_y,\;\; k_+ = \omega/c + k_z,\;\; N^{-1}(\boldsymbol{k}) = \sqrt{2}k_+\omega/c$$

- What is the physical interpretation of $\alpha_\pm(\boldsymbol{k})$?
- It is best seen after field quantization

QED in terms of the RS vector

- In QED the RS vector becomes a field operator

$$\hat{\boldsymbol{F}}(\boldsymbol{r},t) = \int d^3k\, \boldsymbol{e}(\boldsymbol{k}) \sqrt{\frac{\hbar\omega}{(2\pi)^3}} \left(\hat{a}(\boldsymbol{k}) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} + \hat{b}^\dagger(\boldsymbol{k}) e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} \right)$$

- Operators $\hat{a}(\boldsymbol{k})$ and $\hat{b}(\boldsymbol{k})$ annihilate right-handed and left-handed photons with energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$
- Energy, momentum and angular momentum

$$\hat{\mathcal{E}} = \int d^3k\, \hbar\omega \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \right) \quad \hat{\mathcal{P}} = \int d^3k\, \hbar\boldsymbol{k} \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \right)$$

$$\hat{\mathcal{M}}_z = \int d^3k\, \hbar \left(\hat{a}^\dagger \frac{1}{i} \frac{\partial}{\partial_\varphi} \hat{a} + \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \frac{1}{i} \frac{\partial}{\partial_\varphi} \hat{b} - \hat{b}^\dagger \hat{b} \right)$$

Photon states

- A general one-photon state is created from the vacuum by a linear combination of the operators $\hat{a}^\dagger$ and $\hat{b}^\dagger$
  - $\hat{a}_f^\dagger = \int d^3k \left(f_+(\boldsymbol{k})\hat{a}^\dagger(\boldsymbol{k}) + f_-(\boldsymbol{k})\hat{b}^\dagger(\boldsymbol{k}) \right) \quad ||f||^2 = 1$
- Functions $f_\pm(\boldsymbol{k})$ are the probability amplitudes or in other words the wave functions of right-handed and left-handed photons in momentum space
- They correspond to the classical amplitudes $\alpha_\pm(\boldsymbol{k})$
- This relationship is best seen if we use coherent states of the electromagnetic field corresponding to $\hat{a}_f^\dagger$

$$|f\rangle = e^{-\frac{1}{2}\langle\hat{N}\rangle} e^{\sqrt{\langle\hat{N}\rangle}\hat{a}_f^\dagger}|0\rangle \quad \langle\hat{N}\rangle = \text{average photon number}$$

Interpretation of $\alpha_\pm(\boldsymbol{k})$ from QED

- Expectation value of the RS operator in coherent states

$$\boldsymbol{F}_{\rm cl}(\boldsymbol{r},t) = \langle f|\hat{\boldsymbol{F}}(\boldsymbol{r},t)|f\rangle$$
$$= \int d^3k\; \boldsymbol{e}(\boldsymbol{k})\sqrt{\frac{\langle\hat{N}\rangle\hbar\omega}{(2\pi)^3}}\left(f_+(\boldsymbol{k})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} + f_-^*(\boldsymbol{k})e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

- Therefore, apart from a different normalization, the classical amplitude $\alpha_+(\boldsymbol{k})$ can be identified with the wave function of the right-handed photons and $\alpha_-^*(\boldsymbol{k})$ with the wave function of the left-handed photons
- The RS operator has no dispersion in coherent states

$$\langle f|\left(\hat{\boldsymbol{F}}(\boldsymbol{r},t) - \boldsymbol{F}_{\rm cl}(\boldsymbol{r},t)\right)^n|f\rangle = 0$$

Beams with angular momentum

- All beams of EM radiation are made of photons
- With every classical RS vector we can associate the wave functions $f_+(\boldsymbol{k})$ and $f_-(\boldsymbol{k})$ of the photons that characterize this particular wave field
- Examples of the photon wave functions that lead to some well known solutions of Maxwell equation
  - $f_+(\boldsymbol{k}) = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})$ — left-polarized plane wave
  - $f_-(\boldsymbol{k}) = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q})$ — right-polarized plane wave
  - $f_\pm(\boldsymbol{k}) = \delta(k_z - q_z)\delta(k_\perp - q_\perp)e^{iM\varphi}$ — Bessel beams
  - $f_\pm(\boldsymbol{k}) = f(k_\perp, k_z)e^{im\varphi}$ — any beam with $M_z = m \pm 1$
  - $f_-(\boldsymbol{k}) = \delta(k + k_z - k_0)(k - k_z)^{n+m/2}e^{-l(k-k_z)}e^{im\varphi}$ — the so called exact Laguerre-Gauss beam

Wigner function

- The RS vector is a natural starting point to introduce the Wigner function for the photon through the standard definition adopted from quantum mechanics

$$W_{ij}(\mathbf{r},\mathbf{k},t) = \int d^3s\, e^{-i\mathbf{k}\cdot\mathbf{s}}\, F_i(\mathbf{r}+\mathbf{s}/2,t)F_j^*(\mathbf{r}-\mathbf{s}/2,t)$$

$$W_{ij} = w_{ij} + \frac{c}{2i}\epsilon_{ijk}u_k$$

$$\partial_t w_{ij} = -c\epsilon_{ilk}k_l w_{kj} - c\epsilon_{jlk}k_l w_{ki} - \frac{c^2}{2}(\nabla_i u_j + \nabla_j u_i - \delta_{ij}\nabla_k u_k)$$

$$\partial_t u_i = -c\epsilon_{ijk}k_j u_k - \frac{1}{2}(\nabla_j w_{ij} - \nabla_i w_{jj})$$